Top-Bottom Interference for Higgs Production in Gluon Fusion at NLO in QCD

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Abstract

With the discovery of the Higgs boson, it has never been more crucial to make better theoretical predictions of its properties. For this reason, precision physics is highly important to reduce the theoretical uncertainties. The dominant production mechanism for the Higgs boson at the LHC is gluon fusion. Gluon fusion is mediated by a quark loop since the Higgs boson does not have direct coupling to the gluons. Due to its high mass, top quark contributes the most to the cross section for Higgs boson production in gluon fusion. Recently, there has been important advances on the computation of the cross section for gluon fusion in the effective field theory approach where the top quark is integrated out, reducing the theoretical uncertainty below a few percent. With this development, the next important theoretical uncertainty is due to bottom quark effects.

In this thesis we present the computation for top-bottom interference to the Higgs production cross section in gluon fusion at next-to-leading order (NLO) in QCD as an expansion in the bottom quark mass. This serves as the first order in bottom quark mass in a given strong coupling order. There are two pieces to this cross section: double-virtual and real-virtual contributions. For each case, we describe how we computed analytically the matrix elements, as well as master integrals as an expansion in vanishing bottom quark mass. We present the results for the partonic cross section, matrix elements, master integrals.

To obtain the result as an expansion, we use an extensive technology: IBP reductions, differential equations, Mellin Barnes representation, expansion by regions. We determine a basis of 17 and 16 master integrals for double-virtual and real virtual cases respectively, and obtain differential equations of these masters with respect to the bottom quark mass, as an expansion, in canonical form using Moser's algorithm. We solve differential equations for each master in terms of boundary conditions and

determine the boundary conditions by computing necessary regions for the masters. We describe how we perform these calculations with the Mellin Barnes representation method in detail.

Precision physics constitutes one aspect of high energy physics, while developing new physics theories being another. While the former makes accurate predictions for the Standard Model, the latter could explain the deviations between the predictions and experimental results, or could be detected at the colliders directly. In the last part of this thesis, we present the MATHEMATICA package FEYNRULES. FEYNRULES allows one to generate Feynman rules automatically from the vertices of the Lagrangian of a given model, Standard Model, or a new physics model. FEYNRULEShas a new module, the decay module, which enables one to compute the two-body tree level decay rates of a model automatically. The decay width information can then be imported as a Universal Feynrules Output (UFO) to be used in a matrix element generator. The full decay width of a particle is necessary to make a reliable phenomenological analysis with a Monte Carlo generator. Therefore, automating the process of computing decay widths for different benchmark scenarios of a new physics model is desirable.

Zusammenfassung

Nach der Entdeckung des Higgs Bosons ist es wichtiger denn zuvor bessere theoretische Vorhersagen seiner Eigenschaften zu machen. Aus diesem Grund ist Präzisionsphysik hochgeradig wichtig um die theoretischen Unischerheiten zu reduzieren. Der dominante Produktionsmechanismus für Higgs Bosonen am LHC ist die Gluonen Fusion. Gluonen Fusion wird durch eine Quarkschleife vermittelt, da das Higgs Boson nicht direkt an die Gluonen koppelt. Aufgrund seiner Masse, trägt das Top Quark am meissten zum Produktionsquerschnitt durch Gluonen Fusion bei. In der letzten Zeit gab es wichtige Fortschritte in der Berechnung des Wirkungsquerschnittes für Gluonen Fusion in einer effektiven Feldtheorie, bei der das Top Quark ausintegriert wird. Diese Fortschritte haben die theoretischen Unsicherheiten des Wirkungsquerschnittes auf wenige Prozent reduziert. Eine wichtige verbleibende Unsicherheit des Wirkungsquerschnittes wird durch das Bottom Quark verursacht.

In dieser Arbeit präsentieren wir die Rechnung von Beiträgen zum Produktionsquerschnitt in Gluon Fusion in nächst-führender Ordnung (NLO) in QCD, welche von Interferenzen von Top und Bottom Quarks ausgelöst wird. Dies ist der führende Beitrag in der Masse des Bottom Quarks in einer jeden Ordnung in der starken Kopplungskonstanten. Der Streuquerschnitt setzt sich aus zwei Beiträgen zusammen: den zweifach virtuellen (double-virtual), sowie den reel-virtuellen (real-virtual) Beiträgen. In beiden Fällen beschreiben wir die analytische Berechnung der Matrixelemente und Masterintegrale, als Entwicklung in der verschwindenden Masse des Bottom Quarks. Wir präsentieren die Ergebnisse für den partonischen Wirkungsquerschnitt, die Matrixelemente, sowie die Master integrale.

Um diese Ergebnisse als Entwicklung zu berechnen benutzen wir umfangreiche Technologie: IBP Reduktionen, Differentialgleichungen, Mellin-Barnes Darstellungen

und Expansion-by-Regions. Wir bestimmen eine Basis von 17 Masterintegralen, für den double-virtual Beitrag und 16 Masterintegralen für den real-virtual Beitrag. Für diese Masterintegrale leiten wir Differentialgleichungen nach der Bottom Quark masse her und bringen die Entwicklung mittels Mosers Algorithmus in kanonische Form. Wir lösen die Differnetialgleichungen für jedes Masterintegral und bestimmen die benötigten Randbedingungen indem wir die Entsprechenden Regionen der Masterintegrale berechnen. Wir beschreiben im Detail, wie wir diese Regionen mittels Mellin-Barnes Darstellungen berechnen.

Präzisionsphysik ist lediglich ein Aspekt der Hochenergiephysik. Ein wichtiger andere Aspekt ist die Entwicklung von theorien für neue Physik. Die Präzisionsphysik versucht moeglichst exakte Vorhersagen für Beobachtungen zu machen, währen man versucht mögliche Abweichungen der Experimentellen Ergebnisse von den Vorhersagen mittels Modellen für neue Physik zu erklären.

Im letzten Teil dieser Arbeit präsentieren wir das MathematicaPacket FeynRules. FeynRulesermöglicht es automatisch die Feynman Regel aus dem Lagrangian eines gegebenen Models zu bestimmen, sei dies das Standard Modell oder ein Modell neuer Physik. Mittels eines neuen Moduls, dem Zerfallsmodul, ermöglicht es FeynRulesnun auch die Zweikörper Zerfälle auf dem Born Level automatisch zu berechnen. Diese Informatioen über die Zerfallsbreite kann dann mittels des Univeral Feynrules Output in einen Matrixelement Generator importiert werden. Um verlässliche phenomenologische Vorhersagen über instabile Teilchen in Monte-Carlo Simulationen zu machen ist es notwenig die volle Zerfallsbreite des Teilchens zu kennen. Die Automatisierung der Berechnung von Zerfallsbreiten für Modelle neuer Physik ist deshalb ein wichtiger Schritt.

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Declaration

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- Romain Mueller and Deniz Gizem Oeztuerk. On the computation of finite bottom-quark mass effects in Higgs boson production. 2015, [arXiv:1512.0857], to be submitted to JHEP.

- Johan Alwall, Claude Duhr, Benjamin Fuks, Olivier Mattelaer, Deniz Gizem Öztürk, and Chia-Hsien Shen. Computing decay rates for new physics theories with FeynRules and MadGraph 5_aMC@NLO. Comput. Phys. Commun., 197 (2015) 312-323, [arXiv:1402.1178].

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Chapter 1

Introduction

The year 2012 witnessed an amazing discovery: the *discovery of the Higgs boson* by the Large Hadron Collider (LHC) at CERN [1, 2]. The Higgs Boson, which was predicted by Peter Higgs and Francois Englert et al. [3–6], was the final and the most important piece of the Standard Model (SM) that had not been found. The Higgs boson is crucial firstly because it is the only fundamental scalar particle observed in nature so far. The existence of the Higgs field gives rise to Electroweak symmetry breaking, and explains how the SM particles obtain their masses. It is the interaction, or non-interaction, of the Higgs field with the other fields that makes the subsequent quanta of these fields massive or not. From another theoretical perspective, a boson was necessary in order to preserve unitarity in the W-boson scattering, and the Higgs boson serves to solve this problem. Finally, the Higgs boson can open up a door to the discovery of the New Physics, as the Higgs boson might be the particle that connects the SM to the New Physics.

Discovering the Higgs boson was possible due to diligent work both from experimentalists and theorists. Once the Higgs boson had been found, the next step is to determine the properties of it. Discrepancies between the measured results and the theoretical predictions can give us an idea about the nature of the potential New Physics. However, theoretical predictions for cross sections at the LHC suffer from potentially large theoretical uncertainties, which needs to be reduced in order to better distinguish whether a discrepancy is a result of the uncertainties or New Physics. The theoretical uncertainties can arise due to the fact that cross sections are computed in perturbation theory which introduces a dependence on an unphysical scale choice into the result. This scale dependence is reduced by computing higher orders in perturbation theory. Computing higher order terms is not an easy task, and becomes exponentially more difficult as the order increases. Therefore, in the past couple of decades since the prediction of the Higgs boson, there has been substantial effort and progress towards having a greater understanding of the theoretical computations.

The Higgs boson can be produced via various processes. The dominant production mechanism of the Higgs boson at the LHC is via gluon fusion. This process is mediated by a quark loop since the Higgs boson does not have a direct coupling to the gluons. The coupling of the Higgs boson to quarks is proportional to the mass of the quarks. Since the mass of the top quark is substantially larger than that of other quarks, the cross section for this channel is mostly dominated by a top quark loop [7, 8]. The next-to-leading order (NLO) QCD corrections to this process with full top mass dependence is computed in refs. [9–15]. Top quarks can be integrated out since the mass of the Higgs boson is smaller than twice the mass of the top quark. This results in a description of a *Heavy Quark Effective Field Theory* approach where the top quark mass is treated as infinite and the Higgs boson has a direct coupling to the gluons, which constitutes now the leading order (LO) for this process. The NLO QCD corrections for this process in the effective theory which is a one loop calculation, has been computed in the 90s [16]. The computation for the next-to-next-to leading order (NNLO) corrections in QCD was performed by [17–19]. The virtual corrections next-to-next-to-next-to leading order (N3LO) in QCD are calculated in refs. [20, 21]. While the effective theory presents a first approximation to the cross section in the limit of an infinitely heavy top, the subleading corrections that account for the finite top mass have been computed in ref. [22, 23]. In the last few years, there have been great improvements made towards obtaining the corrections to the Higgs cross section beyond NNLO. The remaining N3LO corrections have been computed in the threshold expansion [24–32]. This level of precision reduced the theoretical uncertainty due to scale variation to less than 3% [29, 33]. However, the cross section is subject to other types of uncertainties. One important source of uncertainty is due to quark mass effects. In particular, while corrections to the effective theory due to the finite top quark mass have been computed, there is an important class of corrections that has not been studied extensively. These corrections are due to the finite bottom quark mass.

In particular, corrections coming from a finite bottom-quark mass account for

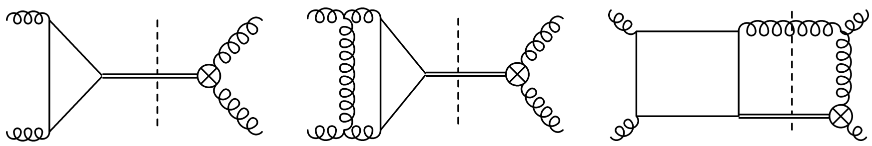


Figure 1.1: Typical contributions of order m_b^2 to the gg-initiated cross-section: Born (left), virtual (middle) and real (right). Curly lines indicate a gluon, plain lines a b-quark, and doubled lines a Higgs boson. The crossed circle indicates the Hgg effective vertex .

around 6% [33] of the total Higgs boson production cross section at leading order and around 4% at NLO [33–35]. This means that one can expect the uncertainty coming from the unknown finite bottom-quark mass effects at NNLO to contribute substantially to the total uncertainty on the inclusive Higgs cross section at the LHC and the computation of such effects is desirable. A review of the current precision for Higgs production in gluon fusion can be found in ref. [33].

In this work we are interested precisely in the bottom quark mass effects on the production of the Higgs boson in gluon fusion in QCD. We will investigate this by computing the inclusive cross section for the so called *top-bottom interference* for production of the Higgs boson via the gluon fusion at NLO (at order α_s^3) as an expansion in bottom quark mass. Top-bottom interference means that the contributions where the gluon fusion is mediated with a bottom quark are interfered these with the same process mediated with top quarks. The way we compute this is to use the effective theory where the top quarks integrated out, and explicitly expand around vanishing bottom quark mass. At a particular order in α_s, contributions that are only due to bottom quark are proportional to square of the Yukawa coupling: y_b^2, and are thus significantly suppressed compared to the interference which is proportional to $y_b y_t$ due to the smallness of the bottom quark mass. Typical diagrams contributing are shown in Figure 1.1. There are two pieces to top-bottom interference: double-virtual and real-virtual contributions. The double virtual matrix elements, i.e. the two loop contribution to this calculation is already performed analytically [8, 12, 14, 36, 37], and real radiation matrix elements have been computed in refs. [38–40]. We presented the first time the analytic result for the interference cross section here [41]. Here in this thesis, we will present an analytic expression of the cross section for the top-bottom interference as an expansion in the bottom quark mass at NLO in QCD following the work we have done [41].

To perform this calculation, we have developed a systematic approach, based on differential equations [42–44] and Mellin-Barnes representations that allows us to obtain an expansion of the interference cross section to in principle any order in the bottom quark mass. This technology will be particularly useful for computing the NNLO contributions to top-bottom interference. A calculation with full mass dependence at this level in perturbation theory is not feasible at this point as the computation is expected to involve elliptic integrals. [45]. Using our method to obtain expansions in the masses the appearance of these functions can be circumvented.

The first part of this thesis will be about the background and methods we used which will set the stage for the calculation. and will consist of the chapters *Theory Overview*, *Feynman Integrals* and *Inclusive Cross Section in QCD*. In the first chapter, we will give a brief introduction to Quantum Chromodynamics (QCD) and the Heavy Quark Effective Field Theory. In Chapter 3, called *Feynman Integrals*, we will discuss various aspects of Feynman integrals such as integration by parts identities (IBPs) and differential equations. Next, we will discuss the methods used to compute Feynman integrals: Mellin Barnes representation and the method of differential equations. In Chapter 4, *Inclusive Cross Section in QCD*, we will describe how to compute the cross section in QCD in perturbation theory together with different ingredients. In the second part we will describe our calculation of the cross section for top-bottom interference in the following chapters: *Cross Section at NLO*, *Real Virtual Contribution* and *Double Virtual Contribution*. While in the first chapter we will present our results for the cross section; in the next two chapters we will explain the different pieces contributing to the cross section and present the corresponding results.

As mentioned earlier, precision predictions for SM Higgs boson are used to determine deviations which could be indicative of New Physics. The next step after reaching a certain theoretical precision, the goal would be to determine what kind of new physics the discrepancies are due to. For this reason, various beyond the Standard Model (BSM) models have been proposed, and still being developed; the predictions of which should then be compared to the experimental results. Proposing a new BSM model require determining the accompanying Lagrangian and the Feynman rules, which are then to be used to make phenomenological predictions. Therefore, it is crucial to automate the first step of this process. The MATHEMATICA package FEYNRULES [46–49] successfully does this, which will be the topic of the last chapter of this thesis.

Part I

Theory

Chapter 2

Theory Overview

2.1 Introduction to QCD

Quantum Chromodynamics (QCD) is a non-abelian gauge theory which describes the strong interactions between quarks and gluons. Historically, M. Gell-Mann [50] and G. Zweig [51] proposed ‘quarks’ as the building blocks of strongly interacting particles, such as protons and neutrons. The discovery of Δ^{++} particle led to the proposal of the existence of another quantum number by Greenberg [52], Nambu and Han [53, 54]: the so called colour charge N_C. The reason for this is that Δ^{++} particle consists of three up-quarks which, in the absence of another quantum number, violates the Pauli Exclusion Principle. By comparing the theoretical and experimentally measured results for the ratio of the cross sections for the following processes N_C could be determined:

$$\frac{\sigma\left(e^+e^- \to \text{hadrons}\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} = N_C \sum e_q^2 = \frac{11}{9} \tag{2.1}$$

resulting in $N_C \sim 3.2$.

QCD is a non-abelian gauge theory with SU(N_C) being the gauge group. The invariance under local gauge transformations requires the existence of gauge fields that are gluons. The Lagrangian for QCD can be written in three parts:

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{Yang-Mills}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{ghosts}} \tag{2.2}$$

$$\mathcal{L}_{\text{Yang-Mills}} = -\frac{1}{2}F^{\mu\nu,a}F_{\mu\nu,a} + \sum_f \bar{\psi}_f^i \left(i\not{D}_{ij} - m_f\delta_{ij}\right)\, \psi_f^j \tag{2.3}$$

where ψ_f is the fermion field, f is the flavor index, and i, j are the color indices for fermions, whereas a is the color index for gluons. $\not{D} = \gamma_\mu D^\mu$ is the covariant derivative in the fundamental representation:

$$(D_\mu)_{ij} = \partial_\mu\delta_{ij} - igA_\mu^a t_{ij}^a \tag{2.4}$$

where t_{ij}^a's are the generators of fundamental representation of SU(3), g the strong coupling. $a = 1, \cdots N_C^2 - 1$ is the color index, A_μ^a is the gauge field. $F_{\mu\nu}^a$ is the field strength tensor for the gluon field:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf^{abc}A_\mu^b A_\nu^c \tag{2.5}$$

The generators form the Lie algebra of the group SU(3):

$$\left[t^a, t^b\right] = if^{abc}t^c \tag{2.6}$$

$$tr\left(t^a t^b\right) = T_R\delta^{ab} \tag{2.7}$$

where the second line indicates the choice of normalization, and f^{abc} are the structure constants, which are related to the structure constants in the adjoint representation, $(T_A)_{bc}^a$, as:

$$(T_A)_{bc}^a = -if^{abc} \tag{2.8}$$

with

$$tr\left(T_A^a T_A^b\right) = C_A\delta^{ab}, \quad C_A = N_C \tag{2.9}$$

$$\sum_c \left(T_A^c T_A^c\right)_{ab} = C_F\delta_{ab}, \quad C_F = \frac{N_C^2 - 1}{2N_C} \tag{2.10}$$

The Lagrangian in Equation 2.3 is invariant under global and local transformations:

$$U = e^{i\theta^a(x)t^a} \tag{2.11}$$

where U is a unitary matrix, an element of SU(3), $\theta^a(x)$ are the gauge parameters. The fields transform under gauge transformations as:

$$\psi_i(x) \to \psi_i'(x) = U_{ij}\psi_j(x) \tag{2.12}$$

$$A_\mu(x) \to A'_\mu(x) = U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}\left(\partial_\mu U(x)\right)U^\dagger(x) \qquad (2.13)$$
$$\text{with} \quad A_\mu = A^a_\mu t^a$$

With the Lagrangian at hand, the propagators and vertices can be determined. However, due to gauge invariance the inverse of the equation of motion for the gluon field does not exist. This is due to gauge invariance, which allows for field configurations that lead to the same physics, which are therefore redundant. For this reason, *gauge fixing* is introduced which eliminates these redundancies:

$$G_\mathrm{F}[A^\xi] = 0 \qquad (2.14)$$

this ensures that the gauge parameter $\xi(x)$ is fixed. With this, the kinetic term of the Lagrangian for the gluon is invertible, therefore the gluon propagator exists. There are different gauge fixing conditions. Here we will use covariant gauges where $G_\mathrm{F} = \partial^\mu A^a_\mu$. Then the gauge fixing Lagrangian, $\mathcal{L}_\mathrm{GF}$, is written as:

$$\mathcal{L}_\mathrm{GF} = -\frac{1}{2\xi}\left(G_\mathrm{F}[A^\xi]\right)^2 = -\frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)^2 \qquad (2.15)$$

The particular choice of the gauge parameter, $\xi = 1$ is called the Feynman-'t Hooft gauge, and this is what we will use in this thesis.

The quantization of the QCD Lagrangian is possible only if one introduces the unphysical ghost fields. The Lagrangian for Faddeev-Popov ghosts is:

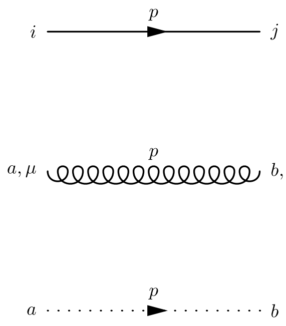
$$\mathcal{L}_\mathrm{ghosts} = \bar{c}^a\left(-\partial^\mu D^{ab}_\mu\right)c^b \qquad (2.16)$$

where $c, \bar{c}$ are the ghost fields, which are anti commuting, and in the adjoint representation.

Feynman Rules for QCD

Here we will present the Feynman rules for QCD. We keep the gauge dependence ξ explicit here, although we will use the Feynman gauge where $\xi = 1$ throughout this thesis. The propagators and interaction vertices are given below in the figure.

For the three-gluon vertex, the momenta are treated as incoming. The spin and



$$= \frac{i\delta_{ij}\left(\not{p}+m\right)}{p^2-m^2+i\epsilon}$$

$$= \frac{-i\delta^{ab}}{p^2+i\epsilon}\left[g^{\mu\nu}-(1-\xi)\frac{p^\mu p^\nu}{p^2}\right]$$

$$= \frac{i\delta^{ab}}{p^2+i\epsilon}$$

Figure 2.1: Propagators in QCD

polarization sums for gluons and fermions that we use are:

$$\sum_s \bar{u}^s(p)u^s(p) = \not{p}+m \tag{2.17}$$

$$\sum_s \bar{v}^s(p)v^s(p) = \not{p}-m \tag{2.18}$$

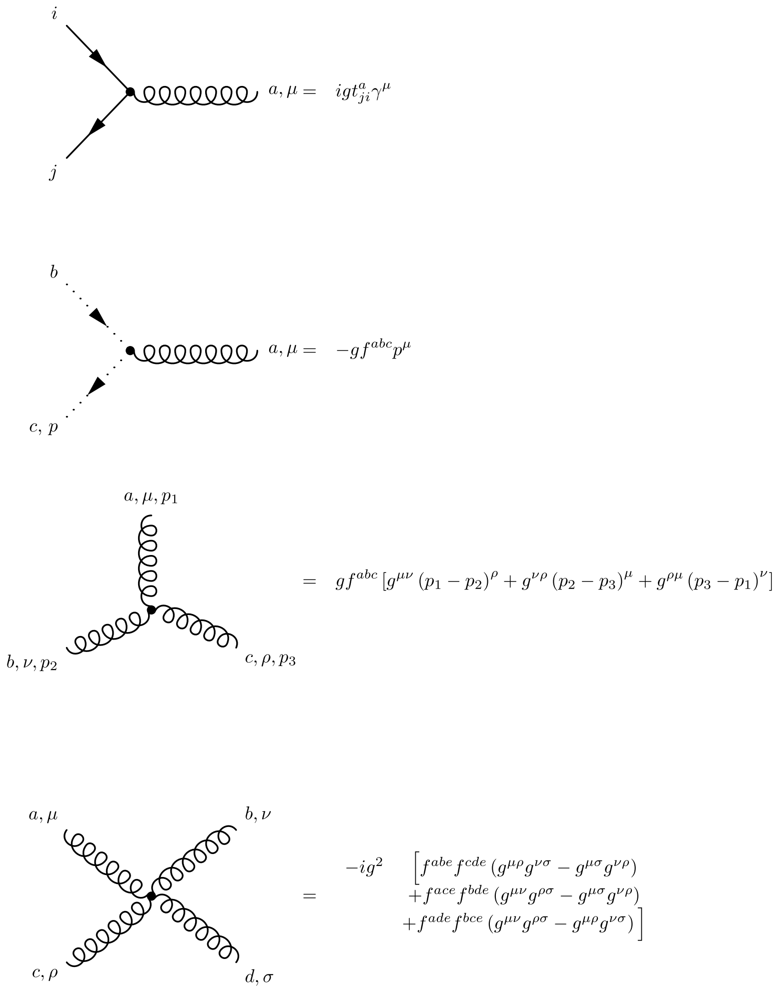
$$\sum_{pols} \epsilon^{\mu *}\epsilon^{\nu} = -\,g^{\mu\nu} \tag{2.19}$$

In addition to these, we have:

- For each fermion and ghost loop, add a multiplicative factor of (-1).
- For each loop with loop momentum k, perform an integration over k: $\int d^dk/(i\pi)^d$, where d is the space-time dimension.
- Take into account the symmetry factors.

2.2 Heavy Quark Effective Field Theory

An effective theory framework is necessary to use when the scales of a theory are separated such that higher scales are irrelevant to the low-energy phenomena. In such a framework, a low-energy approximation is made where the theory is valid up to a certain scale Λ, and the fields heavier than this scale decouple from the theory. Then these fields can be removed from the theory by being 'integrated out', and their effects can then be absorbed to the parameters of the low-energy theory.



$$= \; igt^a_{ji}\gamma^\mu$$

$$= \; -gf^{abc}p^\mu$$

$$= \; gf^{abc}\left[g^{\mu\nu}\left(p_1-p_2\right)^\rho + g^{\nu\rho}\left(p_2-p_3\right)^\mu + g^{\rho\mu}\left(p_3-p_1\right)^\nu\right]$$

$$= \; -ig^2 \Big[f^{abe}f^{cde}\left(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}\right) + f^{ace}f^{bde}\left(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho}\right) + f^{ade}f^{bce}\left(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}\right)\Big]$$

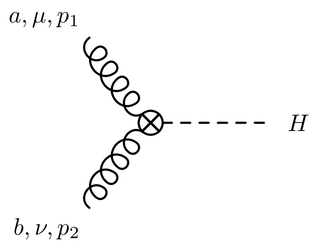
Figure 2.2: Vertices in QCD

In this work, we are interested in the top-bottom interference contribution to the Higgs production in gluon fusion. The gluon fusion channel is mediated by a quark loop since Higgs boson does not have direct coupling to gluons. Since the top quark is heavier than the Higgs boson, an effective field theory approach can be used where top quark is integrated out from the Higgs-quark-antiquark vertex. In this formalism the effective Lagrangian describing the Higgs-gluon-gluon interaction is written in terms of an effective operator. Together with the Yukawa interactions for n_l light quarks, the Lagrangian reads:

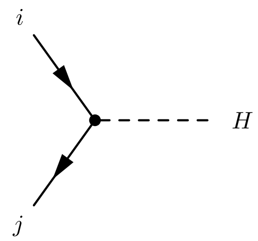
$$\mathcal{L}_{\text{eff}} = -\frac{1}{4v} c_H \, F^a_{\mu\nu} F^{\mu\nu;a} H \tag{2.20}$$

$$\mathcal{L}_{\text{Yukawa}} = \sum_f^{n_l} \frac{m_f}{v} \, H \, \bar{\psi}_f \, \psi_f \tag{2.21}$$

where c_H is the Wilson coefficient [55–60] which should be computed, $F^a_{\mu\nu}$ is the gluon field strength tensor, H is the physical Higgs field, v is the vacuum expectation value of the Higgs field, ψ_f are the fermion fields. The Hgg and Hff vertices is given by:



$$= -i\delta^{ab} c_H \Big(- g^{\mu\nu} \, (p_1 \cdot p_2) + p_1^\nu p_2^\mu \Big)$$



$$= -i \, \frac{m_f}{v} \, \delta_{ij}$$

Figure 2.3: Higgs interactions with quarks in the heavy quark effective theory. Hgg effective vertex is depicted as the Feynman diagram at the top, while Hff vertex is displayed below. Feynman rule for each vertex is written on thhe right hand side of the graph.

Effective theory approach resulted in a good approximation compared to the full result at leading order (LO) for gluon fusion [16]. The derivation of the effective theory formalism includes matching the relevant parameters of the full theory to that of effective theory, and computation of Wilson coefficient. This is beyond the scope of this thesis and we refer [61] for details.

Chapter 3

Feynman Integrals

When calculating scattering amplitudes in quantum field theories perturbative approach can be used when the coupling is weak enough. Then the amplitude can be expanded around this coupling, and one can calculate each term independently until the desired order if possible. Feynman found out that processes in quantum field theories can be expressed diagrammatically which we call as *Feynman Diagrams*. Feynman diagrams come in handy when calculating amplitudes because one can easily generate them according to the interactions allowed in the theory being worked on, and then apply certain rules, called *Feynman Rules*, to them to translate them back into analytic forms.

As the order in the perturbative series increases the complexity increases as well: virtual particles are allowed to be created and annihilated due to uncertainty principle, and the corrections due to these will contribute at higher orders. These particles can have any momentum which cannot be measured therefore one has to integrate over all possible values during a computation. These integrals are called *Feynman Integrals*, and they are expressed as loop diagrams with the momentum to be integrated being the *loop momentum*. The general form of a Feynman integral in four dimensions is:

$$I = \int \left(\prod_{j=1}^{L} d^4 k_j \right) \frac{\mathcal{N}\left(\{p_i, k_j\}\right)}{D_1^{\nu_1} D_2^{\nu_2} \ldots D_n^{\nu_n}} \tag{3.1}$$

where k_j, with $1 \leq j \leq L$, are the loop momenta, $\nu_i \in \mathbb{Z}$ where $1 \leq i \leq n$ are integer exponents, and p_i are the external momenta. The numerator $\mathcal{N}\left(\{p_i, k_j\}\right)$ is a polynomial of the scalar products of external and loop momenta, whereas D_i are the denominators of propagators which have the following form:

$$D_i = q_i^2 - m_i^2 + i0 \tag{3.2}$$

where q_i is the momentum flowing through the propagator which is a linear combination of external momenta and loop momenta, m_i is the mass of the particle and the small imaginary part $i0$ denotes the Feynman prescription.

Feynman integrals have remarkable properties that we will briefly introduce in Sections 3.1 and 3.2. Together with these properties, computing Feynman integrals is a big part of obtaining theoretical predictions on observables such as cross sections or calculating scattering amplitudes in general. Feynman integrals can be computed by several techniques. The successful methods for the evaluation of two and three loop integrals include *Feynman Parametrization*, *Mellin Barnes Representation Method*, and *The Method of Differential Equations*, which we will also use extensively in combinations in this thesis. These will be the topic of Section 3.3.

3.1 Introduction

3.1.1 Divergences

Feynman integrals can have two types of divergences: ultraviolet (UV) and infrared (IR) divergences. These divergences correspond to different regions of the loop integration as well as the specific kinematic configurations of the process under consideration. UV divergences appear when $k \to \infty$, with k being the loop momentum, therefore correspond to high energy (short distance) interactions. The IR divergences, which can be two types: *soft* or *collinear* divergences. Soft divergences occur when $k \to 0$ or when the momentum of an external massless particle varnishes. Collinear divergences on the other hand occur when two massless particles become collinear.

Let us give an example of a Feynman integral which has a UV singularity. Consider the massive tadpole integral:

$$\int \frac{d^4k}{[k^2 - m^2]} \underset{k \to \infty}{\sim} \frac{k^4}{k^2} \sim k^2 \tag{3.3}$$

where k is the loop momentum, whereas m is the mass of the fermion. For convenience, we did not write the $+i0$ prescription explicitly. From simple power counting we see that this integral is divergent as $k \to \infty$. And this is indeed the case [62].

In general it is common to encounter divergences in quantum field theories. The nature of particular quantum field theories allows us to deal with these divergences by redefining the parameters of their Lagrangian or the wavefunctions and absorbing the infinities into those. This procedure is called *Renormalization*, and these quantum field theories are said to be *Renormalizable*. The Standard Model (SM) Lagrangian is renormalizable **ref** and in particular in SM, the UV divergences are absorbed into parameters via this renormalization procedure. IR divergences cancel for IR safe observables such as inclusive cross sections, which we will be considering in this thesis.

Before one can absorb UV divergences or cancel IR divergences, however, one needs to parameterise these infinities by introducing a regulator. This parametrisation is called *Regularization*, with the new parameter being the regularization parameter. Once divergences are removed, the physical limit should be taken so that the result does not depend on the regularization parameter.

Several methods of regularization have been developed. One of them is the momentum cut off procedure. In this case the upper bound of the integral is cut at a certain scale Λ which renders the integral finite. This is allowed since in principle the theory at hand would be valid only up to a certain scale. Another method is Pauli-Villar regularization where massless propagators are given a fictitious mass term. The problem with these procedures is that they do not preserve all the symmetries of the integral, such as Lorentz or gauge symmetries. Nowadays the most used regularization method when calculating Feynman integrals is *Dimensional Regularization*. This is also what we use to regularize the integrals we calculate for this thesis, which will be described in the following sections.

Once the regularization is applied the integral can be calculated. There are several methods to perform the loop integrations. Below we will focus on *The Method of Mellin Barnes Representation* and *The Method of Differential Equations*.

3.1.2 Dimensional Regularization

Introduced by t'Hooft dimensional regularization is a very powerful regularization method where the regularization parameter is the space-time dimension d. The integrals which are divergent in 4 dimensions are evaluated at $d = 4 - 2\epsilon$ dimensions with d being now a non integer, where they are well behaved. The prescription to work on dimensional regularization is:

$$\int d^4 k \quad \rightarrow \quad \int e^{\gamma_E \epsilon} \frac{d^d k}{i\pi^{d/2}} \tag{3.4}$$

where γ_E is the Euler-Mascheroni constant, and $e^{\gamma_E \epsilon}/\left(i\pi^{d/2}\right)$ is the normalisation factor.

Once the integral is evaluated in d dimensions, a Laurent expansion in ϵ around zero can be performed:

$$\mathrm{I} = \sum_k \mathrm{I}_k \epsilon^k = \mathrm{I}_{k_0} \epsilon^{k_0} + \mathrm{I}_{k_0+1} \epsilon^{k_0+1} + \mathrm{I}_{k_0+2} \epsilon^{k_0+2} + \dots \tag{3.5}$$

where $k, k_0 \in \mathbb{Z}$ and $k_0 \leq k$. The information about infinities are then encapsulated as the singularities in ϵ, i.e. when $k_0 < 0$. After adding all contributions to the inclusive cross section, the result should not depend on the newly introduced parameter ϵ, and the physical limit $\epsilon \rightarrow 0$ should be taken at the end of the calculation. With dimensional regularization both the UV and IR divergences are regularized. It must be noted that with this method it is not possible to distinguish between the poles that originate from UV divergences and the poles originate from IR divergences since they all appear as poles in ϵ.

In dimensional regularisation, we perform the integrals in d dimensions, however the action should be kept dimensionless. This leads to the introduction of an auxiliary mass scale to the theory. For this reason, the gauge coupling should be rescaled as:

$$g \rightarrow g\mu^{\epsilon} \tag{3.6}$$

In the following we will keep the scale μ implicit, and only write it explicitly when we present our results for the cross section for top-bottom interference for the Higgs production in gluon fusion at NLO.

We will describe which methods we can use to calculate Feynman integrals in d dimensions in the next sections, but before let us introduce special functions that appear in the analytic expressions of Feynman integrals.

3.1.3 Mathematical Interlude

There has been remarkable observations made concerning Feynman integrals in the last decades.It turns out that the analytic expressions for Feynman integrals do not just consist *any* number or functions, but of special classes of numbers and functions, some of which include *Multiple Zeta Values* and *Multiple Polylogarithms* [63]. These functions posses certain analytic structures and algebraic properties [64–66] which reveal important properties about Feynman integrals. In this section we will give a very brief introduction to these special classes of functions and numbers that will appear when calculating Feynman integrals.

For Feynman integrals with many different scales more complicated functions such as *Elliptic Functions* [45] may appear. The algebraic relations of elliptic functions are not yet known, therefore dealing with multiple scales in certain multi-loop integrals remains still a challenge. As mentioned in the Introduction, this is one of the reasons why in this thesis we apply a certain procedure to expand our integrals in the small mass scale. This procedure may not be necessary for a two loops, two to one process, however it will be very useful for an application to three loops of the same process; which is not analytically calculated yet. We will explain the method in Sections 3.3 and 6.2.

Basics

Let us start with the *Classical Polylogarithms*, $\mathrm{Li}_n(z)$. Classical Polylogarithms are a generalisation of logarithms which are defined recursively:

$$\log(z) = \int_1^z \frac{dt}{t} \tag{3.7}$$

$$\mathrm{Li}_n(z) = \int_0^z \frac{dt}{t}\,\mathrm{Li}_{n-1}(t), \quad \text{with} \quad \mathrm{Li}_1(z) = -\log(1-z) \tag{3.8}$$

Another important mathematical object that we encounter in the analytic expressions of Feynman integrals is the *Zeta Values*. Zeta values are the *Riemann Zeta Function*, ζ_n evaluated at integer values, for $n > 1$. One of the representations of ζ_n is a sum:

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n}, \quad \Re(n) > 1 \tag{3.9}$$

Zeta values can also be expressed as polylogarithms evaluated at one, which is a consequence of the sum representation of the polylogarithms. For $|z| \leq 1$:

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \quad n > 1 \tag{3.10}$$

$$\zeta_n = \mathrm{Li}_n(1) \tag{3.11}$$

Another important point is that for positive even integers $2n$, ζ_{2n} is proportional to π^{2n}:

$$\zeta_{2n} = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2\,(2n)\,!} \tag{3.12}$$

where B_{2n} are the Bernoulli numbers which are defined as:

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \tag{3.13}$$

With these definitions at hand, we can express the zeta values ζ_2 and ζ_4 as:

$$\zeta_2 = \frac{\pi^2}{6}, \quad \zeta_4 = \frac{\pi^4}{90} \tag{3.14}$$

A generalization of classical polylogarithm is the *Nielsen's Polylogarithm*:

$$S_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)!\,p!} \int_0^1 \frac{dt}{t} \ln^{n-1}(t)\, \ln^p(1-zt) \tag{3.15}$$

For $p = 1$, it reduces to classical polylogarithm: $S_{n,1}(z) = \mathrm{Li}_{n+1}(z)$.

And finally, we give the definition for the *Gamma function*:

$$\Gamma\,(n) = \int_0^{\infty} x^{n-1}\, e^{-x}\, dx \tag{3.16}$$

where n can be a complex number with $\Re(n) > 0$. A useful property that we will use is:

$$\Gamma(n+1) = n\,\Gamma(n) \tag{3.17}$$

Multiple Polylogarithms

Multiple polylogarithms (MPLs) are generalisations of classical polylogarithms, with now multi variables; which are also defined recursively via iterated integrals [63, 67]:

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t)\,, \quad a_i, z \in \mathbb{C} \tag{3.18}$$

$$\text{with} \quad G(;z) = 1 \tag{3.19}$$

where a_i's can be either constants or functions of other variables. Note that the integral is divergent for $z = a_1$. The number of elements of the vector $\vec{a} = (a_1, \cdots, a_n)$, namely n is called the *weight* of the multiple polylogarithm. In special cases the multiple polylogarithms become the ordinary logarithms or classical polylogarithms, and in particular up to weight three all MPLs can be expressed in terms of classical polylogarithms and ordinary logarithms:

$$G\left(\vec{0}_n;z\right) = \frac{1}{n!}\,\log^n z \tag{3.20}$$

$$G(\vec{a}_n;z) = \frac{1}{n!}\log^n\left(1-\frac{z}{a}\right) \tag{3.21}$$

$$G\left(\vec{0}_{n-1},a;z\right) = -\mathrm{Li}_n\left(\frac{z}{a}\right) \tag{3.22}$$

MPLs have several properties [64, 67, 68] some of which are extensively used in calculating multi loop integrals in scattering amplitudes [69–73]. MPLs are equipped with *shuffle product* [74], that can be explained in the following way. Consider two sets of letters (a,b) and (w,y), which will appear as the indices of MPLs later, and assume that we would like to write down all possible sets from these letters. The requirement is that, when we do so, the ordering of the letters from each starting set should be kept intact:

$$\begin{aligned}(a,b) \sqcup\!\sqcup\, (w,y) = {} & (a,b,w,y) + (a,w,b,y) + (w,a,b,y) \\ & + (a,w,y,b) + (w,a,y,b) + (w,y,a,b)\end{aligned} \tag{3.23}$$

where we can see that a always appears before b; and w always appears before y. For MPLs a simple example of using shuffle product would be:

$$G(a,b;z)\; G(c;z) = G(a,b,c;z) + G(a,c,b;z) + G(c,a,b;z) \tag{3.24}$$

Shuffle algebra can be very useful to make simplifications when calculating Feynman integrals. More complicated examples can be found in [75].

Harmonic Polylogarithms (HPLs) [76–79] is a special class of MPLs, with $a_i \in \{-1, 0, 1\}$. They were introduced to High Energy Physics literature by Remiddi and Varmaseren [76], and are denoted with H rather than G. HPLs are related to MPLs in the following way:

$$H(\vec{a};z) = (-1)^p\, G(\vec{a};z) \tag{3.25}$$

where p is the number of elements in $\vec{a}$ that are 1.

There are codes available with implementations of HPLs for numerical evaluation and analytical simplifications [77–81].

Hypergeometric Functions

Gauss' hypergeometric function is defined as:

$${}_2F_1\Big(\{a_1,a_2\};\{b_1\};z\Big) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n}{(b_1)_n} \frac{z^n}{n!} \tag{3.26}$$

$${}_3F_2\Big(\{a_1,a_2,a_3\};\{b_1,b_2\};z\Big) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n}{(b_1)_n (b_2)_n} \frac{z^n}{n!} \tag{3.27}$$

where $(l)_n$, with $l \in \{a_1, a_2, a_3, b_1, b_2\}$, is the Pochhammer symbol:

$$(l)_n = \frac{\Gamma(l+n)}{\Gamma(l)} = l(l+1)\dots(l+n-1) \tag{3.28}$$

The Euler integral representation is:

$${}_2F_1\Big(\{a_1,a_2\},\{b_1\};z\Big) = \frac{\Gamma(b_1)}{\Gamma(a_2)\,\Gamma(b_1-a_2)} \int_0^1 dt\, t^{a_2-1}(1-t)^{b_1-a_2-1}(1-zt)^{-a_1} \tag{3.29}$$

The generalization of Hypergeometric function in series represantation is:

$${}_pF_q\Big(\{a_1\dots a_p\};\{b_1\dots b_q\};z\Big) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^n}{n!}$$

$$= \frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)} \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{3.30}$$

$(a_i)_n$ is again the Pochhammer symbol.

A useful MATHEMATICA package to expand Hypergeometric functions is HYPEXP [82, 83], which we also use to expand literature results of two loop masters to compare with our results.

3.2 Relations between Feynman Integrals

Feynman integrals have remarkable relations among them. In this section we will describe some of these relations: *Integration by Parts Identities*, *Dimensional Shift Identities* and *Differential Equations*. Although these relations may be used for many different purposes, in practical multi-loop calculations we use the first class of relations in order to reduce the number of integrals that have to be evaluated. The second class is used in this thesis as a check of the analytical calculations of the Feynman integrals that we compute. The last one, on the other hand, is an elegant way to evaluate Feynman integrals without directly computing them. The details will be presented in the following sections.

3.2.1 Integration by Parts Identities

Calculating amplitudes becomes a more difficult task as the number of loops/legs increase since the complexity also increases immensely. One reason is simply that the number of diagrams contribute increases from one perturbative order to the next. For example while at leading order there are only two diagrams that appear for the Higgs production in gluon fusion in full theory, this number increases to twelve at two loops and is at the level of hundreds at three loops respectively. Squaring these diagrams result in non-trivial amount of terms, which after the tensor reduction is performed leads to for three loops for instance thousands of integrals that need to be calculated.

Fortunately some of these integrals can be related to others by means of special identities. By deriving first these identities one can refrain from calculating all of the integrals, but can compute only a few and write down the remaining ones in terms of others. This way the integrals can be systematically eliminated from being required to be calculated until one is left with certain number of integrals that cannot be reduced anymore. These integrals are called *Master Integrals.* Master integrals form a basis such that the rest of the integrals can be written as a linear combination of these basis integrals.

One class of these identities is Integration by Parts (IBP) identities [43, 84–86]. IBP relations are a result of the extension of the Gauss' theorem to d dimensions, which states that an integration over the loop momentum of a total derivative with respect to the same loop momentum gives zero in dimensional regularization:

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{\partial}{\partial k_\mu} f^\mu(k, \ldots) = 0 \tag{3.31}$$

where f^μ is a function of scalar products of external momenta and loop momentum. Upon acting with the differential operator on the integrand and evaluating the result one can express a Feynman integral in terms of other integrals with rational coefficients.

In order to see how the IBP's work we will start by giving a simple example. Consider a Feynman integral $\mathrm{I}[\nu_1, \nu_2, \nu_3]$ in d dimensions with three distinct massless propagators with arbitrary integer exponents: ν_1, ν_2 and ν_3; with k being the loop momentum, where as p_1 and p_2 are momenta of the external particles which we take as massless:

$$\mathrm{I}\left[\nu_1, \nu_2, \nu_3\right] = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3}} \tag{3.32}$$

where $D_1 = k^2$, $D_2 = (k - p_1)^2$ and $D_3 = (k - p_1 - p_2)^2$. Now we will use the operator $\partial/\partial k_\mu$ to act on the integrand. To have a Lorentz scalar quantity after taking the total derivative we should multiply this integral by a vector, which in this case three available: k^μ, p_1^μ and p_2^μ. In this example we will choose this vector to be k^μ to obtain:

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{\partial}{\partial k_\mu} \left(\frac{k^\mu}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3}} \right) = 0 \tag{3.33}$$

Evaluating the left hand side we get:

$$\left(\left(\frac{\partial}{\partial k_\mu} k^\mu \right) \frac{1}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3}} + k^\mu \frac{\partial}{\partial k_\mu} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3}} \right) \tag{3.34}$$

$$= \left(\frac{d}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3}} + \frac{-2\nu_1 k^2}{D_1^{\nu_1+1} D_2^{\nu_2} D_3^{\nu_3}} + \frac{-2\nu_2 (k^2 - k\cdot p_1)}{D_1^{\nu_1} D_2^{\nu_2+1} D_3^{\nu_3}} + \frac{-2\nu_3 (k^2 - k\cdot p_1 - k\cdot p_2)}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3+1}} \right) \tag{3.35}$$

$$= \left(\frac{d - 2\nu_1 - \nu_2 - \nu_3}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3}} - \frac{\nu_2}{D_1^{\nu_1-1} D_2^{\nu_2+1} D_3^{\nu_3}} - \frac{\nu_3}{D_1^{\nu_1-1} D_2^{\nu_2} D_3^{\nu_3+1}} + \frac{s_{12}\nu_3}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3+1}} \right) \tag{3.36}$$

$$= (d - 2\nu_1 - \nu_2 - \nu_3)\, \mathrm{I}\,[\nu_1, \nu_2, \nu_3] - \nu_2\, \mathrm{I}\,[\nu_1 - 1, \nu_2 + 1, \nu_3] \tag{3.37}$$

$$-\nu_3\, \mathrm{I}\,[\nu_1 - 1, \nu_2, \nu_3 + 1] + s_{12}\, \nu_3\, \mathrm{I}\,[\nu_1, \nu_2, \nu_3 + 1] \tag{3.38}$$

where from second line to the third we wrote the scalar products in terms of denominators: $k^2 = D_1$, $2k \cdot p_1 = D_1 - D_2$ and $2k \cdot p_2 = D_2 - D_3 + s_{12}$, with $s_{12} = 2p_1 \cdot p_2$ being the center of mass energy. Here it should be noted that the scalar products are uniquely expressed through the propagators at hand. A set of propagators in terms of which all the scalar products can be uniquely written, is called a *topology*.

The last equation we obtained is an IBP identity which states a relation between integrals that belong to the same topology but have denominators with different exponents. For our example, we could also repeat the same steps by multiplying our integral by vectors p_1^μ and p_2^μ at the beginning rather than k^μ which would give us two more identities.

For a topology of seven denominators, which is the case for top-bottom interference for Higgs production in gluon fusion at NLO since this is a two loop process, we have in total six identities:

$$\mathrm{I}\,[\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7] = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{(i\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5} D_6^{\nu_6} D_7^{\nu_7}} \tag{3.39}$$

$$\int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{\partial}{\partial l_\mu} \left(\frac{q^\mu}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5} D_6^{\nu_6} D_7^{\nu_7}} \right) = 0 \tag{3.40}$$

where $l^\mu \in \{k_1^\mu, k_2^\mu\}$ and $q^\mu \in \{k_1^\mu, k_2^\mu, p_1^\mu, p_2^\mu\}$.

We will present in the following one of the IBP identities. Consider the following topology which appears in the real virtual case:

$$D_1 = (-p_1 - p_2 - p_3)^2 - m_H^2, \tag{3.41}$$
$$D_2 = p_3^2, \tag{3.42}$$
$$D_3 = k^2 - m_b^2 \tag{3.43}$$
$$D_4 = (k - p_1 - p_2 - p_3)^2 - m_b^2, \tag{3.44}$$
$$D_5 = (k - p_1)^2 - m_b^2, \tag{3.45}$$
$$D_6 = (k - p_1 - p_3)^2 - m_b^2 \tag{3.46}$$
$$D_7 = (p_2 + p_3)^2 \tag{3.47}$$

where D_1 the Higgs propagator with m_H being the Higgs mass, whereas p_1 and p_2 are the momenta of the incoming gluons, p_3 is the momentum of the outgoing gluon that is cut, and k is the loop momentum. In total we have 6 IBP identities for this structure as explained above where the first one is:

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{d^d p_3}{i\pi^{d/2}} \frac{\partial}{\partial k_\mu} \left(\frac{k^\mu}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5} D_6^{\nu_6} D_7^{\nu_7}} \right) = 0 \tag{3.48}$$

which when evaluated reads:

$$\begin{aligned}
&(d - 2\nu_3 - \nu_4 - \nu_5 - \nu_6)\, \mathrm{I}\left[\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7\right] \\
&+ \left(\nu_4 z - 2\, m_b^2\, \nu_4\right) \mathrm{I}\left[\nu_1, \nu_2, \nu_3, \nu_4 + 1, \nu_5, \nu_6, \nu_7\right] \\
&+ \left(-2m_b^2\, \nu_6 + \nu_6 z - \nu_6\right) \mathrm{I}\left[\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6 + 1, \nu_7\right] - 2m_b^2\, \nu_3 \mathrm{I}\left[\nu_1, \nu_2, \nu_3 + 1, \nu_4, \nu_5, \nu_6, \nu_7\right] \\
&-2m_b^2\, \nu_5 \mathrm{I}\left[\nu_1, \nu_2, \nu_3, \nu_4, \nu_5 + 1, \nu_6, \nu_7\right] + \nu_4 \mathrm{I}\left[\nu_1 - 1, \nu_2, \nu_3, \nu_4 + 1, \nu_5, \nu_6, \nu_7\right] \\
&-\nu_4 \mathrm{I}\left[\nu_1, \nu_2, \nu_3 - 1, \nu_4 + 1, \nu_5, \nu_6, \nu_7\right] - \nu_5 \mathrm{I}\left[\nu_1, \nu_2, \nu_3 - 1, \nu_4, \nu_5 + 1, \nu_6, \nu_7\right] \\
&+\nu_6 \mathrm{I}\left[\nu_1 - 1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6 + 1, \nu_7\right] + \nu_6 \mathrm{I}\left[\nu_1, \nu_2 - 1, \nu_3, \nu_4, \nu_5, \nu_6 + 1, \nu_7\right] \\
&-\nu_6 \mathrm{I}\left[\nu_1, \nu_2, \nu_3 - 1, \nu_4, \nu_5, \nu_6 + 1, \nu_7\right] - \nu_6 \mathrm{I}\left[\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6 + 1, \nu_7 - 1\right] = 0
\end{aligned} \tag{3.49}$$

The rest of the IBP identities are found in a similar manner and will not be displayed here. The exponents ν_1, ν_2 and ν_3 in principle can take any integer value and one can obtain relations for many different integrals from a single IBP identity. At multi-loop calculations the procedure to write down the IBP's to reduce the integrals by hand is quite tedious, therefore these relations should be dealt with systematically. *Laporta Algorithm* [86] is a Gauss elimination method, that can be used to reduce the integrals systematically by means of IBP identities. Several reduction programmes that implements Laporta Algorithm and the generalization of it are available now, such as AIR [87], FIRE [88, 89], REDUZE [90–92], and LiteRed [93]. For this thesis we performed the reduction a private C++ code[1].

It is important to note that the choice of the basis for the master integrals is arbitrary. From the point of view of calculating an integral brute force, e.g. performing the loop integrations directly, one can decide which basis to choose depending on the complexity of the integrals. This can be specified in reduction programs, i.e. allowing the program to choose the integrals with propagators with minimum amount of exponents. However the choice of the basis, although arbitrary, requires more effort if one would like to use *The Method of Differential Equations* to evaluate the integrals. The reason for this will become clear in Section 3.3.3.

3.2.2 Dimensional Shift Identities

Dimensional Shift Identities, also known as *Dimensional Recurrence Relations* (DRR), are relations between Feynman integrals that are in different space-time dimensions, d [94–96]. Here we will present a DRR formula for loop integrals, which can be generalised to phase space integrals. The idea is to make a change of variables from the integration variable which is the loop momenta to kinematic invariants s_{ij} in terms of loop momenta:

$$\mathrm{I}(d) = \int \left(\prod_{i=1}^{L} \frac{d^d k_i}{i\pi^{d/2}} \right) \frac{1}{D_1^{\nu_1} \dots D_n^{\nu_n}} \tag{3.50}$$

$$= \frac{\mu^L \pi^{-LE/2 - L(L-1)/4}}{\Gamma\left((d-E-L+1)/2, \dots, (d-E)/2\right)} \int \left(\prod_{i=1}^{L} \prod_{j=1}^{L+E} ds_{ij} \right) \times \frac{G\left(k_1, \dots, k_L, p_1, \dots, p_E\right)^{(d-E-L-1)/2}}{G\left(p_1, \dots, p_E\right)^{(d-E-1)/2}} \frac{1}{D_1^{\nu_1} \dots D_n^{\nu_n}} \tag{3.51}$$

with $s_{ij} = k_i \cdot q_j$, where k_i indicates the loop momenta and q_j can either be the external momenta or the loop momenta. The Jacobian resulting from the change of variables is expressed as a Gram determinant $G\left(\{s_{ij}\}\right)$, where:

$$G\left(p_1, \cdots, p_E\right) = \begin{vmatrix} p_1^2 & \cdots & p_E^2 \\ \vdots & \ddots & \vdots \\ p_1 \cdot p_E & \cdots & p_E^2 \end{vmatrix} \tag{3.52}$$

[1] We thank Bernhard Mistlberger who provided this code.

As seen from the formula above the dependence on d is only in the exponent of the Gram determinant. This means that if we write down the integral on the left hand side in $d+2$ dimensions and relate this back to the original one the relation will be in terms of Gram determinant up to an overall factor:

$$\mathrm{I}\left(d+2\right)=\frac{\left(2\mu\right)^{L}}{\left(d-E-L+1\right)G\left(p_{1},\ldots,p_{E}\right)}G\left(k_{1},\ldots,k_{L},p_{1},\ldots,p_{E}\right)\,I(d) \tag{3.53}$$

The right hand side involves an integral in d dimensions, but also scalar products due to Gram determinant. So it needs to be reduced. The result will be an expression of our integral in $d+2$ dimensions in terms of master integrals in d dimensions.

DRR's provide a good check for the computation of dimensionally regularized multi-loop Feynman integrals. After calculating a Feynman integral in d dimensions using the techniques described in the previous sections, we can shift the dimension to $d+2$ in the analytic expression we find to obtain an analytic expression for the integral in $d+2$ dimensions. Separately we can take the relation above 3.53 and perform the reduction to write a second expression for it. We can then compare these two analytic expressions to check if they match, which will be an indication of the correctness of our direct analytical calculation.

It is important to note that dimensional shift identities hold region by region: i.e. if one region is missed in a calculation of a Feynman integral the relations would still hold. Therefore a further check is still required. In our case the cancellation of the poles in ϵ when obtaining the cross section was a powerful check.

3.3 Methods to Compute Feynman Integrals

3.3.1 Feynman Parametrization

Feynman parametrization is one of the tricks to compute a Feynman integral. The idea is to introduce extra parameters to bring the structure of the integral to a certain form that will make the computation easier: namely the denominators are combined into a single term by means of these parameters, which are constrained by momentum conservation. The drawback is that there are now additional integrals over the new parameters to be performed. In particular, Feynman observed that

$$\frac{1}{D_1 D_2} = \int_0^1 dx_1 \int_0^1 dx_2 \frac{\delta\,(1 - x_1 - x_2)}{\left[x_1 D_1 + x_2 D_2\right]^2} \tag{3.54}$$

Now consider a set of n denominators that appear in a loop integral, $D_1, D_2, \cdots D_n$. We can generalize the above trick for two denominators to n denominators with arbitrary exponents as the following:

$$\frac{1}{D_1^{\nu_1}\, D_2^{\nu_2}\, \ldots D_n^{\nu_n}} = \left(\prod_i^n \int_0^1 \frac{dx_i\, x_i^{\nu_i - 1}}{\Gamma\,(\nu_i)} \right) \frac{\Gamma\,(\nu_1 + \ldots + \nu_n)\, \delta\,(1 - x_1 \ldots - x_n)}{\left[x_1 D_1 + x_2 D_2 + \ldots + x_n D_n\right]^{N_\nu}} \tag{3.55}$$

where $N_\nu = \nu_1 + \cdots \nu_n$ and x's are the new parameters that are introduced. The delta function is due to momentum conservation.

In order to illustrate how the integration is performed, let us give a concrete example. Consider the one loop massless bubble Feynman integral. This integral will appear in the cross section results for top-bottom interference in Chapter 7. For this integral, we introduce two parameters, the same amount as the number of propagators, x_1 and x_2 as the following:

$$B(p^2) = \int \frac{e^{\epsilon\gamma_E} d^d k}{i\pi^{d/2}} \frac{1}{\left[k^2 + i0\right] \left[(k-p)^2 + i0\right]} \tag{3.56}$$

$$= \int_0^1 dx_1 \int_0^1 dx_2 \int \frac{e^{\epsilon\gamma_E} d^d k}{i\pi^{d/2}} \frac{\delta\,(1 - x_1 - x_2)}{\left[k^2\, x_1 + \left((k-p)^2 \right) x_2 \right]^2} \tag{3.57}$$

$$= \int_0^1 dx_1 \int_0^1 dx_2 \int \frac{e^{\epsilon\gamma_E} d^d k}{i\pi^{d/2}} \frac{\delta\,(1 - x_1 - x_2)}{\left[k^2\, (x_1 + x_2) - 2k\, p\, x_2 + p^2\, x_2\right]^2} \tag{3.58}$$

$$= \int_0^1 dx_1 \int \frac{e^{\epsilon\gamma_E} d^d k}{i\pi^{d/2}} \frac{1}{\left[k^2 - 2k\, p\, (1 - x_1) + p^2\, (1 - x_1)\right]^2} \tag{3.59}$$

$$= \int_0^1 dx_1 \int \frac{e^{\epsilon\gamma_E} d^d k}{i\pi^{d/2}} \frac{1}{\left[(k - p\, (1 - x_1))^2 - p^2\, (1 - x_1)^2 + p^2\, (1 - x_1)\right]^2} \tag{3.60}$$

$$= \int_0^1 dx_1 \int \frac{e^{\epsilon\gamma_E} d^d k}{i\pi^{d/2}} \frac{1}{\left[k^2 - (-p^2\, x_1\, (1 - x_1))\right]^2} \tag{3.61}$$

$$= \int_0^1 dx_1 \int \frac{e^{\epsilon\gamma_E} d^d k}{i\pi^{d/2}} \frac{1}{\left[k^2 - \Delta\right]^2} \tag{3.62}$$

From the third to fourth line we integrated over the variable x_2 by using the delta function. From fourth to fifth line we shifted the loop momentum as $k \to k+p\,(1-x_1)$ to remove the linear terms in k. Now we can use the following identity to perform the integration over k:

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{1}{[k^2-\Delta]^n} = -\,(-1)^n\, \frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \tag{3.63}$$

In our case $\Delta = \left(-p^2\, x_1\, (1-x_1)\right)$. Applying this identity in our integral we get:

$$e^{\epsilon\gamma_E} \int_0^1 dx_1 \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Gamma(2)} \left(\frac{1}{(-p^2\, x_1\, (1-x_1))}\right)^{2-\frac{d}{2}} \tag{3.64}$$

$$= \int_0^1 dx_1\, e^{\epsilon\gamma_E}\Gamma(\epsilon)\, \left(-p^2\right)^{-\epsilon} [x_1\, (1-x_1)]^{-\epsilon} \tag{3.65}$$

$$= e^{\epsilon\gamma_E}\Gamma(\epsilon)\,(-s)^{-\epsilon} \int_0^1 dx_1\, x_1^{-1+(1-\epsilon)}\, (1-x_1)^{-1+(1-\epsilon)} \tag{3.66}$$

$$= e^{\epsilon\gamma_E}\Gamma(\epsilon)\,(-s)^{-\epsilon}\; B\,(1-\epsilon, 1-\epsilon) \tag{3.67}$$

where $s = p^2$ is the center of mass energy and we set $d = 4-2\epsilon$. In the last line we brought the integral into such a form that it is the *Beta function*. Now we can expand this result in ϵ up to $\mathcal{O}\left(\epsilon^2\right)$ to get:

$$B(p^2) = \left(\frac{1}{\epsilon} + 2 - \log(-s) + \left(4 - 2\log(-s) + \frac{1}{2}\log^2(-s) - \frac{1}{2}\zeta_2\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)\right) \tag{3.68}$$

where γ_E is the Euler-Mascheroni constant. As we expected, the integral is divergent and the divergence appears as a pole in ϵ.

In general, an L loop Feynman integral with n propagators can be expressed as:

$$\begin{aligned} \mathrm{I}[\nu_1, \cdots, \nu_n] = \Gamma(N_\nu) \int \prod_{i=1}^{n} dx_i\, x_i^{\nu_i - 1}\, \delta\left(1 - \sum_{i=1}^{n} x_i\right) \\ \times \int dk_1 \ldots dk_L \left[\sum_{j,l=1}^{L} k_j\, M_{jl}\, k_l - 2\sum_{j=1}^{L} k_j\, Q_j + J\right]^{-N_\nu} \end{aligned} \tag{3.69}$$

$$
= \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - dL/2\right)}{\prod_{i=1}^{n}\Gamma\left(\nu_i\right)} \int_0^1 \prod_{i=1}^{n} dx_i\, x_i^{\nu_i - 1}\, \delta\left(1 - \sum_{i=1}^{n} x_i\right) \frac{\mathcal{U}^{N_\nu - d(L+1)/2}}{\mathcal{F}^{N_\nu - dL/2}} \tag{3.70}
$$

where k_i are the loop momenta, $N_\nu = \nu_1 + \ldots + \nu_n$. M is an $L \times L$ matrix, Q is a vector. In the first line, the introduction of M, Q and J provides the 'completion of the square' for the loop momenta to achieve the form in Equation 3.63. In the second line, the functions $\mathcal{U}$ and $\mathcal{F}$ are the Symanzik polynomials that are expressed as:

$$
\mathcal{U} = \det(M), \tag{3.71}
$$

$$
\mathcal{F} = \det(M)\left[\sum_{i,j=1}^{L} Q_i M_{ij} Q_j - J\right] \tag{3.72}
$$

Another useful tool when using Feynman parameters representation is the *Cheng-Wu theorem* [97]. Cheng-Wu theorem states that the formula in 3.70 holds for the following delta function:

$$
\delta\left(1 - \sum_{i\in\sigma} x_i\right) \tag{3.73}
$$

where σ is a subset of $n = 1, \ldots L$, if the boundary of the integration over the variables is changed to zero to infinity. In practical calculations of Feynman integrals involving many Feynman parameters, one uses Cheng Wu theorem to eliminate one of the Feynman parameters by setting into 1. Then we can use the following formula to integrate over a Feynman parameter x:

$$
\int_0^\infty dx\, x^a \left(A + x\, B\right)^b = A^{1+a+b}\, B^{-1-a}\, \frac{\Gamma\left(1+a\right)\Gamma\left(-1-a-b\right)}{\Gamma\left(-b\right)} \tag{3.74}
$$

In the rest of this thesis, we will use Cheng Wu theorem extensively when calculating Feynman integrals.

3.3.2 The Method of Mellin Barnes Representation

The Mellin Barnes method is a powerful tool to calculate Feynman integrals, especially the ones with massive propagators. Since in this thesis we tackle massive integrals it is useful to review the procedure for using Mellin Barnes representation here. Then we will give a concrete example where we apply Mellin Barnes representation together with the Feynman parametrization.

One of the identities, first introduced in the concept of Feynman integrals in ref. [98], is:

$$\frac{1}{(1+A)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz A^{z}\Gamma(\lambda+z)\Gamma(-z) \tag{3.75}$$

where z is the new complex integration variable. This formula can be used to factor two terms in a sum at the expense of introducing an integration to be performed. The procedure to calculate the integral in Equation 3.75 is based on the Cauchy's theorem. According to Cauchy's theorem an integral along a contour in the complex plane is equal to the sum of its residues:

$$\int_{C} f(z)dz = 2\pi i \sum_{i} \mathrm{Res}_{z_i} f \tag{3.76}$$

where the function f analytic in z. The Res_{z_i} denotes the residues of f at points $z = z_i$. The contour needs to be chosen such that the poles of $\Gamma(-z+\ldots)$ are separated from the poles of $\Gamma(+z+\ldots)$ and will be closed at infinity. For the integral 3.75 we choose the contour to be at $-1 < \Re(z) < 0$ and close it at infinity to the right. The complex plane for z, together with the positions of the poles, is depicted in Figure 3.1. Then, we take the residues at points $z = 0, 1, 2, \cdots$ to obtain:

$$\frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}(2\pi i)\sum_{n=0}^{\infty}\frac{(-1)^{-n}}{n!}\Gamma(\lambda+n)A^{n} \tag{3.77}$$

In the following we will explain how we can use Mellin Barnes representation to calculate the Feynman integrals.

Feynman Integrals with Massive Propagators

In practical calculations the Mellin Barnes representation is very useful when one has already applied Feynman Parametrization to a given Feynman integral and have several Feynman parameters and additional scales to deal with. In the case where the propagators are massive the extra scale adds another level of complexity to the problem. With the use of Mellin Barnes representation one can factor out the mass from the denominator of the propagator which is in the form of Equation 3.75. We

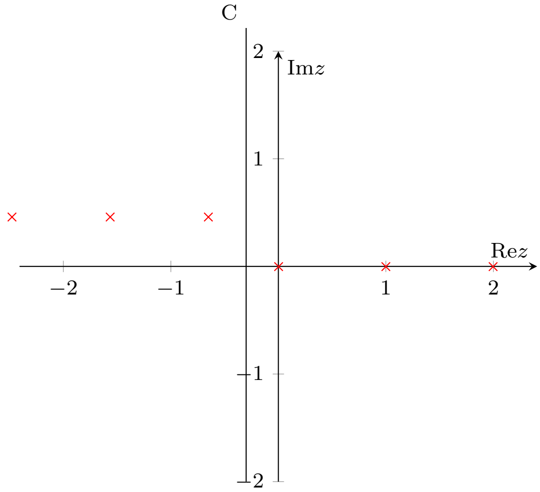


Figure 3.1: The complex plane for the Mellin Barnes variable z in Equation 3.75. The contour lies between $-1 < \Re(z) < 0$ such that the poles from $\Gamma(-z)$ are separated from the ones from $\Gamma(\lambda + z)$. The poles are depicted as crosses. The ones that are to the left of the contour are from $\Gamma(\lambda + z)$ and appears at $z = -\lambda, -\lambda - 1, -\lambda - 2, \ldots$.

shall demonstrate this by considering a massive triangle integral in the following and apply the Feynman parametrization to it together with Mellin Barnes method. For simplification we drop the $i0$ prescription and assume that it is implied. Consider:

$$\mathrm{Tri}\left(s, m^2\right) = e^{\epsilon\,\gamma_E} \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{\left[k^2 - m^2\right]\left[(k+p_1)^2 - m^2\right]\left[(k+p_1+p_2)^2 - m^2\right]} \tag{3.78}$$

where k is the loop momentum to be integrated over and p_1 and p_2 are the external momenta. Since propagators have masses this integral is IR-finite, and by simple power counting we expect that it will be also UV-finite as $\epsilon \to 0$. Now we can apply Feynman parametrization. We introduce three new parameters, x_1, x_2 and x_3 as:

$$\begin{aligned} e^{\epsilon\,\gamma_E} \int \frac{d^d k}{i\pi^{d/2}} \int_0^1 & dx_1 dx_2 dx_3\, \delta\left(1 - x_1 - x_2 - x_3\right) \\ & \times \left[\left(k^2 - m^2\right) x_1 + \left((k+p_1)^2 - m^2\right) x_2 + \left((k+p_1+p_2)^2 - m^2\right) x_3\right]^{-3} \end{aligned} \tag{3.79}$$

$$= e^{\epsilon\,\gamma_E} \int \frac{d^dk}{i\pi^{d/2}} \int_0^1 dx_1 dx_2 dx_3\, \delta\left(1 - x_1 - x_2 - x_3\right) \left(x_1 + x_2 + x_3\right)^{-3}$$
$$\times \left[k^2 + 2k\, \left(p_2 x_3 + p_1(x_2 + x_3)\right) \left(x_1 + x_2 + x_3\right)^{-1} - m^2 + s x_3 \left(x_1 + x_2 + x_3\right)^{-1}\right]^{-3} \tag{3.80}$$

This time we did not immediately integrate over one of the Feynman parameters with the use of the delta function, which will become clear later. Now we can shift the loop momenta as $k \to k - \left(p_2 x_3 + p_1(x_2 + x_3)\right) \left(x_1 + x_2 + x_3\right)^{-1}$ to complete the square and use the identity in Equation 3.63 to integrate over the loop momenta. After some algebra we get:

$$-e^{\epsilon\,\gamma_E}\, \Gamma\left(1 + \epsilon\right) \int_0^1 dx_1 dx_2 dx_3\, \delta\left(1 - x_1 - x_2 - x_3\right) \left(x_1 + x_2 + x_3\right)^{2\epsilon - 1}$$
$$\times \left(m^2 \left(x_1 + x_2 + x_3\right)^2 - s x_1 x_3\right)^{-\epsilon - 1} \tag{3.81}$$

Now we can use the Cheng-Wu theorem to set one of the variables to one, which we will choose as x_1, and change the integration boundaries of the remaining integrals to from zero to infinity:

$$- e^{\epsilon\,\gamma_E}\, \Gamma\left(\epsilon + 1\right) \int_0^\infty dx_2 dx_3\, \left(1 + x_2 + x_3\right)^{2\epsilon - 1} \times \left(m^2 \left(1 + x_2 + x_3\right)^2 - s x_3\right)^{-\epsilon - 1} \tag{3.82}$$

$$= - e^{\epsilon\,\gamma_E} \int_{-i\infty}^{+i\infty} dz_0\, m^{2z_0} \left(-s\right)^{-1 - z_0 - \epsilon}\, \Gamma\left(-z_0\right)\, \Gamma\left(1 + z_0 + \epsilon\right) \int_0^\infty dx_2\, dx_3\, x_3^{-1 - z_0 - \epsilon}$$
$$\times \left(1 + x_2 + x_3\right)^{-1 + 2\epsilon + 2z_0} \tag{3.83}$$

In the last line we factored out the expression and isolated the mass term by using Equation 3.75 at the expense of introducing a Mellin Barnes variable z_0 to be integrated. Now we have the form of 3.74 so we can easily integrate over the variables x_2 and x_3:

$$- e^{\epsilon\,\gamma_E} \int_0^\infty dx_3 \int_{-i\infty}^{+i\infty} dz_0\, \frac{m^{2z_0} \left(-s\right)^{-1 - z_0 - \epsilon}\, \Gamma\left(-z_0\right) \Gamma\left(-2z_0 - 2\epsilon\right) \Gamma\left(1 + \epsilon + z_0\right)}{\Gamma\left(1 - 2z_0 - 2\epsilon\right)}$$
$$x_3^{-1 - \epsilon - z_0} \left(x_3 + 1\right)^{2(z_0 + \epsilon)} \tag{3.84}$$

$$= - e^{\epsilon\,\gamma_E} \int_{-i\infty}^{+i\infty} dz_0 \frac{m^{2z_0} \left(-s\right)^{-1 - z_0 - \epsilon}\, \Gamma\left(-z_0\right) \Gamma\left(-\epsilon - z_0\right)^2 \Gamma\left(1 + \epsilon + z_0\right)}{\Gamma\left(1 - 2z_0 - 2\epsilon\right)} \tag{3.85}$$

Now we are only left with the integration over z_0. We choose the integration contour at $-1 < \Re(z_0) < 0$ to separate the poles of $\Gamma(-z_0 + \cdots)$ from that of $\Gamma(z_0 + \cdots)$, and close it to the right. This means we need to take the residues which are on the right side of the contour, which are

- Due to $\Gamma(-z_0)$: the residues are created at points $z_0 = n_0$, where n_0 is a positive integer. Recall from Equation 3.77 that normally one has to sum up all the residues as a series:

$$-\sum_{n=0}^{\infty} \frac{(-1)^{-n} m^{2n} (-s)^{-1-n-\epsilon} \Gamma(-n-\epsilon)^2 \Gamma(1+n+\epsilon)}{n!\,\Gamma(1-2n-2\epsilon)} \tag{3.86}$$

  However since in our case the mass scale has a dependence in the Mellin Barnes variable, we would like to terminate the series at certain order, say m^0. Then we have:

$$-\frac{(-s)^{-1-\epsilon}\Gamma(-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \tag{3.87}$$

- Due to $\Gamma(-\epsilon - z_0)$: the residues are created at points $z_0 = n_0 + \epsilon$. Notice the ϵ dependence of the residues, which will result in a scale $m^{-2\epsilon}$:

$$\begin{aligned} -\sum_{n=0}^{\infty} & \frac{(-1)^{-2n} m^{2n-2\epsilon} (-s)^{-1-n} \Gamma(-n+\epsilon)\Gamma(1+n)}{(n!)^2\Gamma(1-2n)} \\ & \times \left(\log(m^2) - \log(-s) + 2\psi^{(0)}(1-2n) - \psi^{(0)}(1+n) - \psi^{(0)}(-n+\epsilon)\right) \end{aligned} \tag{3.88}$$

  Again we take the residues such that we get an expression up to order m^0:

$$m^{-2\epsilon}(-s)^{-1}\Gamma(\epsilon)\left(-\gamma_E + \log\left(-\frac{m^2}{s}\right) - \psi^0(\epsilon)\right) \tag{3.89}$$

Adding these two contributions, we obtain the following result for the massive triangle integral:

$$\begin{aligned} \text{Tri} = e^{\epsilon\gamma_E}\Bigg(& -\frac{(-s)^{-1-\epsilon}\Gamma(-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \\ & + m^{-2\epsilon}(-s)^{-1}\Gamma(\epsilon)\left(-\gamma_E + \log\left(-\frac{m^2}{s}\right) - \psi^0(\epsilon)\right) + \mathcal{O}(m^2)\Bigg) \end{aligned} \tag{3.90}$$

The very important point to note here is that the scale, m^2, is nicely factored out in Equation 3.85 and its exponents are determined by the values that the z takes when we pick the residues. This means that by picking up the residues originating from different gamma functions, we could obtain exponents with different ϵ dependence for the mass terms, such as $m^{-\epsilon+a}$, $m^{-2\epsilon+a}$, m^a, with a being an integer. We call these *scalings*. In general, for a generic Feynman integral with $m \to 0$ we can write:

$$\mathrm{I}(m^2, s, \epsilon) = \sum_n m^{n\epsilon} f_n(m^2, s, \epsilon) \tag{3.91}$$

where $n \in \mathbb{Z}$, and $f_n(m^2, s, \epsilon)$ are the coefficients of different scalings.

With Mellin Barnes method, we can automatically obtain different scalings for a given integral. This crucial fact will be the foundation of how we calculate the boundary conditions that appear in real virtual contributions to top-bottom interference for Higgs production via gluon fusion at NLO. We will present more difficult examples to illustrate how we obtain different scalings from Mellin Barnes integrations will be given in Section 6.2.

For completeness, here we give the results for massive bubble and tadpole integrals that we also computed as well:

$$\begin{aligned}\mathrm{Bub}\left(s, m^2\right) = e^{\epsilon\,\gamma_E} \Bigg(& - m^{2-2\epsilon}\,\frac{2\Gamma\left(-1+\epsilon\right)}{s} \\ & + (-s)^{-\epsilon} \left(\frac{\Gamma\left(1-\epsilon\right)^2 \Gamma\left(\epsilon\right)}{\Gamma\left(2-2\epsilon\right)} + m^2\,\frac{\Gamma\left(-\epsilon\right)^2 \Gamma\left(1+\epsilon\right)}{s\,\Gamma\left(-2\epsilon\right)} \right) + \mathcal{O}(m^4) \Bigg)\end{aligned} \tag{3.92}$$

$$\mathrm{Tad}\left(s, m^2\right) = -e^{\epsilon\,\gamma_E}\, m^{2-2\epsilon}\,\Gamma\left(-1+\epsilon\right) \tag{3.93}$$

3.3.3 The Method of Differential Equations

First introduced by Kotikov [99] in the context of Feynman integrals and developed by Gehrmann and Remiddi [43], *Differential Equations* [42, 44, 69] is a great tool to learn more about the properties of Feynman integrals, as well as to compute them. In this section we will briefly describe how the differential equations work and how these equations are used to compute Feynman integrals.

Being functions of external invariants and internal masses, Feynman integrals satisfy differential equations with respect to these parameters. Consider a Feynman integral $\mathrm{I}\left(s_{ij}, \cdots, r, \epsilon\right)$, where $s_{ij} = p_i \cdot p_j$, with p_i and p_j being two external momenta, and r is the small parameter that is related to the bottom quark mass, which in this case we will assume $r = m_b^2/s$. We can write a first order differential equation with respect to r of the form:

$$\frac{\partial}{\partial r}\mathrm{I}\left(s_{ij}, \ldots, r, \epsilon\right) = a(r, \epsilon)\,\mathrm{I}\left(s_{ij}, \ldots, r, \epsilon\right) + \sum_i b_i(r, \epsilon)\,\mathrm{I}'_i\left(s_{ij}, \ldots, r, \epsilon\right) \tag{3.94}$$

where $a(r, \epsilon)$ and $b_i(r, \epsilon)$ are rational coefficients; the ellipses indicate other scales that the Feynman integrals might depend on and $\mathrm{I}'_i\left(s_{ij}, \ldots, r, \epsilon\right)$ are a list of master integrals that are in the same topology, but have different exponents than the original integral $\mathrm{I}\left(s_{ij}, \ldots, r, \epsilon\right)$. The solution to the homogeneous part of this equation is given by:

$$\mathrm{I}\left(s_{ij}, \ldots, r, \epsilon\right) = e^{\int dr' a(r', \epsilon)} \tag{3.95}$$

whereas the full solution can be written of the form:

$$\mathrm{I}\left(s_{ij}, \ldots, r, \epsilon\right) = e^{\int dr' a(r', \epsilon)}\left(\int dr'\, e^{-\int dr''\, a(r'', \epsilon)} + C\right) \tag{3.96}$$

where C is the integration constant.

Writing a differential equation for each master integral we can express the set of all differential equations for our basis in a matrix form:

$$\partial_r \mathbf{I} = \mathrm{A}\left(r, \epsilon\right)\mathbf{I} \tag{3.97}$$

where now $\mathbf{I}$ is a vector of n master integrals and $\mathrm{A}\left(r, \epsilon\right)$ is an $n \times n$ matrix with entries being the functions of the respective scale and ϵ. For convenience, we used the short hand notation $\partial_r = \frac{\partial}{\partial r}$.

The methods to compute Feynman integrals that we discussed so far, namely Feynman parametrization and Mellin Barnes Representation rely on the direct calculation of the integration over the loop momenta. For two loop integrals with two mass scales, as in the top-bottom interference at NLO, the computation can be tedious with these methods. The Method of Differential Equations allows us to compute Feynman integrals without having to perform the loop integrations explicitly. The idea is to derive the differential equations for master integrals with respect to external momenta or internal masses and then solve these equations by using certain boundary conditions. These boundary conditions correspond to specific kinematic limits of the integrals at hand, therefore a basis of integrals will be expressed only in terms of a set of boundary conditions. These boundary conditions are Feynman integrals that are simpler than the original integrals themselves. In particular, they appear as the coefficients of the *scalings* of the integral. We know from the method of expansion by regions [100, 101] that the asymptotic expansion of a Feynman integral has the following form:

$$\mathbf{I}(r) = \sum_{\alpha} r^{-\alpha\epsilon}\, \mathbf{I}^{\alpha}(r) \tag{3.98}$$

where now different α values indicate different scalings. Our aim is to obtain these scalings from the solution of the differential equations themselves. Then, what is left is to compute the boundary conditions.

The usual strategy during this procedure is to seek to solutions in the form of an expansion in ϵ in the first place [102, 103]. For a coupled system of differential equations, finding the solutions is usually hard. Recently there has been crucial developments made in this respect [44, 104–106]. It has been conjectured that it is possible to rotate the basis of master integrals such that the system can be put into a canonical form where the dimensional regularization parameter ϵ is factored out from the kinematical dependence of the matrix $\mathrm{A}(r,\epsilon)$ [44]:

$$\partial_r \mathbf{I}'(r,\epsilon) = \epsilon\, \mathrm{A}'(r)\, \mathbf{I}'(r,\epsilon) \tag{3.99}$$

with the following change of basis of integrals:

$$\mathbf{I}(r,\epsilon) = \mathrm{T}(r,\epsilon)\, \mathbf{I}'(r,\epsilon) \tag{3.100}$$

$$\mathrm{A}'(r) = \mathrm{T}^{-1}(r,\epsilon)\, (\mathrm{A}(r,\epsilon) - \partial_r)\, \mathrm{T}(r,\epsilon) \tag{3.101}$$

where $\mathrm{T}(r,\epsilon)$ is the transformation matrix. Changing the basis in this way therefore simplifies the structure of the system of differential equations, and allows us to find solutions easier. Once this form is reached, we can solve the system order by order in ϵ.

Obtaining an expansion in the bottom quark mass

In our case for the top-bottom interference, we would like to have a result as an expansion in bottom quark mass. The approach to obtaining differential equations and their solutions as an expansion in an external parameter has already been applied, see for example refs. [24, 107, 108]. We use a slight variant of the algorithm that is proposed by Lee [106], to put the differential equations in a form that allows a simple solution as a power series in the bottom quark mass. Now following ref. [41] we will describe the method. In particular, when expanded in vanishing r, the differential equations should take the following form such that a solution as an expansion in the mass is obtained easily:

$$\partial_r \mathbf{I}(r,\epsilon) = \left(\frac{\mathrm{R}\,(\epsilon)}{r} + \mathrm{A}_0\,(\epsilon) + \dots\right) \mathbf{I}(r,\epsilon) \tag{3.102}$$

$$= \left(\frac{\mathrm{R}\,(\epsilon)}{r} + \sum_{n=0}^{\infty} \mathrm{A}_n\,(\epsilon)\,r^n\right) \mathbf{I}(r,\epsilon) \tag{3.103}$$

where we require that the eigenvalues of the matrix $\mathrm{R}\,(\epsilon)$ is proportional to ϵ. Ignoring higher terms in the expansion on the right hand side, the solution read:

$$\mathbf{I}(r,\epsilon) = \exp\left(\int^r dr'\,\frac{\mathrm{R}\,(\epsilon)}{r'}\right) \mathbf{BC}(\epsilon) = r^{\mathrm{R}(\epsilon)}\,\mathbf{BC}(\epsilon) \tag{3.104}$$

where $\mathbf{BC}(\epsilon)$ are the boundary conditions. This solution precisely contains the structure of the scalings that the integrals will have. This means we can rewrite $\mathbf{I}$ of the form:

$$\mathbf{I}(r,\epsilon) = r^{\mathrm{R}(\epsilon)}\,\tilde{\mathbf{I}}(r,\epsilon) \tag{3.105}$$

where now the dependence in higher orders in r are encapsulated in $\tilde{\mathbf{I}}(r,\epsilon)$. We will now look for a solution for $\tilde{\mathbf{I}}(r,\epsilon)$. Plugging this back into Equation 3.103 we obtain a system of differential equations for $\tilde{\mathbf{I}}(r,\epsilon)$:

$$\partial_r \tilde{\mathbf{I}}(r,\epsilon) = r^{-\mathrm{R}(\epsilon)} \left(\sum_{n=0}^{\infty} \mathrm{A}_n\,(\epsilon)\,r^n\right) r^{\mathrm{R}(\epsilon)}\,\tilde{\mathbf{I}}(r,\epsilon) = \left(\sum_{n=0}^{\infty} \tilde{\mathrm{A}}_n\,(r,\epsilon)\,r^n\right) \tilde{\mathbf{I}}(r,\epsilon) \tag{3.106}$$

where we defined $\tilde{\mathrm{A}}_n\,(r,\epsilon) = r^{-\mathrm{R}}\mathrm{A}_n\,(\epsilon)\,r^{\mathrm{R}}$. The solution can be written as:

$$\tilde{\mathbf{I}}(r,\epsilon) = \mathbf{BC}(\epsilon) + \int_0^r dr_1 \left(\sum_{n=0}^{\infty} \tilde{\mathrm{A}}_n\,(r_1,\epsilon)\,r_1^n\right) \tilde{\mathbf{I}}(r_1,\epsilon) \tag{3.107}$$

Plugging this back to itself, we obtain an expansion in r as Dyson series:

$$\tilde{\mathbf{I}}(r,\epsilon) = \sum_{n=0}^{\infty} \mathrm{D}_n(r,\epsilon)\,\mathbf{BC}(\epsilon) \tag{3.108}$$

where the coefficients of the Dyson series are given by:

$$\begin{aligned}
\mathrm{D}_0(r,\epsilon) &= \mathbb{1} \\
\mathrm{D}_1(r,\epsilon) &= \int_0^r dr_1\,\tilde{\mathrm{A}}_0\,(r_1,\epsilon) \\
\mathrm{D}_2(r,\epsilon) &= \int_0^r dr_1\,\tilde{\mathrm{A}}_0\,(r_1,\epsilon) \int_0^{r_1} dr_2\,\tilde{\mathrm{A}}_0\,(r_2,\epsilon) + \int_0^r dr_1\,r_1\,\tilde{\mathrm{A}}_1\,(r_1,\epsilon) \\
&\ \vdots
\end{aligned} \tag{3.109}$$

Now combining equations 3.105 and 3.108 the solution to our system of differential equations can be written of the form:

$$\mathbf{I}\,(r,\epsilon) = \mathrm{F}\,(r,\epsilon)\,\mathbf{BC}(\epsilon) \tag{3.110}$$

where $\mathbf{I}$ is the vector of our master integrals, and $\mathbf{BC}(\epsilon)$ are the boundary conditions. The matrix $\mathrm{F}\,(r,\epsilon)$ is given by:

$$\mathrm{F}\,(r,\epsilon) = \mathrm{T_{tot}}\,(r,\epsilon)\ r^{\mathrm{R}(\epsilon)} \sum_{n=0}^{\infty} \mathrm{D}_n\,(r,\epsilon) \tag{3.111}$$

where $\mathrm{T_{tot}}$ is the transformation matrix used to achieve the specific form in Equation 3.102.

Now we can explain what $\mathrm{T_{tot}}$ is in more detail. There are three stages to constructing this transformation matrix:

- The first step is the removal of spurious singularities in r. To be more specific, when the differential equations are obtained, they could be of the form:

$$\partial_r\,\mathbf{I}'' = \left(\frac{\mathrm{A}''\,(\epsilon)}{r^2} + \frac{\mathrm{R}''\,(\epsilon)}{r} + \ldots\right)\mathbf{I}'' \tag{3.112}$$

  If the residue matrix $\mathrm{A}''\,(\epsilon)$ is nilpotent, then this highest singularity is indeed spurious and can be removed [109]. We then rotate the master integrals by applying a transformation, $\mathrm{T}_{\mathrm{rank}}(r,\epsilon)$:

$$\mathbf{I}''(r,\epsilon) = \mathrm{T}_{\mathrm{rank}}(r,\epsilon)\,\mathbf{I}'(r,\epsilon) \tag{3.113}$$

such that we obtain:

$$\partial_r\,\mathbf{I}' = \left(\frac{\mathrm{R}'(\epsilon)}{r} + \ldots\right)\mathbf{I}' \tag{3.114}$$

- The second step is diagonalizing the matrix $\mathrm{R}'(\epsilon)$. This step is a preparation for factoring out the ϵ, which will be explained in the next step.

  Jordan decomposition allows us to put a matrix in diagonal form, or if this is not possible, in Jordan normal form. For a square matrix $\mathrm{R}'(\epsilon)$ with eigenvalues $\lambda_1,\ldots,\lambda_n$ there exists a similarity transformation $\mathrm{S}[\mathrm{R}']$ such that:

$$\mathrm{J}\left[\mathrm{R}'\right] = \mathrm{S}\left[\mathrm{R}'\right]^{-1}\mathrm{R}'\,\mathrm{S}\left[\mathrm{R}'\right] \tag{3.115}$$

  where

$$\mathrm{J}[\mathrm{R}'] = \begin{pmatrix} \mathrm{J}_{a_1}^{d_1} & & \\ & \ddots & \\ & & \mathrm{J}_{a_n}^{d_n} \end{pmatrix}, \qquad \text{where} \quad \mathrm{J}_a^d = \begin{pmatrix} a & 1 & & \\ & a & \ddots & \\ & & \ddots & 1 \\ & & & a \end{pmatrix}, \tag{3.116}$$

  with $a_i \in \{\lambda_1,\ldots,\lambda_n\}$ and $d_i \in \mathbb{N}$. The matrix $\mathrm{J}[\mathrm{R}']$ is the Jordan normal form of R' and the d_i-dimensional square matrices $\mathrm{J}_{a_i}^{d_i}$ are its Jordan blocks.

- Lastly, we normalize the eigenvalues of the matrix $\mathrm{R}'(\epsilon)$ such that they are proportional to ϵ. Then ϵ can be factored out.

  This is accomplished by shifting the eigenvalues of $\mathrm{R}'(\epsilon)$. Let us first write down $\mathrm{S}[\mathrm{R}']$ in the following way:

$$\mathrm{S}\left[\mathrm{R}'\right] = (\vec{u}_1,\ldots,\vec{u}_m)\,, \quad \mathrm{S}\left[\mathrm{R}'\right]^{-1} = (\vec{v}_1,\ldots,\vec{v}_m)^T \tag{3.117}$$

  where the vectors $\vec{u}_i$ and $\vec{v}_i$ are generalized eigenvectors of the matrix $\mathrm{R}'(\epsilon)$. We can shift i-th eigenvalue of $\mathrm{R}'(\epsilon)$ from λ_i to $\lambda_i + 1$ using the transformation:

$$\mathrm{T}_i(r,\epsilon) = \mathbb{1} + \frac{1-r}{r}\,\vec{u}_i\vec{v}_i^{\mathrm{T}} \tag{3.118}$$

with

$$\mathrm{T}_i^{-1}(r,\epsilon) = \mathbb{1} - (1-r)\,\vec{u}_i\vec{v}_i^{\mathrm{T}} \tag{3.119}$$

Then the total transformation is obtained as:

$$\mathrm{T}_{\mathrm{norm}}(r,\epsilon) = \prod_i \mathrm{T}_i(r,\epsilon) \tag{3.120}$$

$$\mathbf{I}'\,(r,\epsilon) = \mathrm{T}_{\mathrm{norm}}\,(r,\epsilon)\;\mathbf{I}\,(r,\epsilon) \tag{3.121}$$

such that

$$\partial_r\,\mathbf{I}(r,\epsilon) = \left(\frac{\epsilon\,\mathrm{R}}{r} + \ldots\right)\mathbf{I}(r,\epsilon) \tag{3.122}$$

Then we can write the total transformation as $\mathrm{T}_{\mathrm{tot}} = \mathrm{T}_{\mathrm{rank}}\,\mathrm{T}_{\mathrm{norm}}$.

Further details to our method of solving differential equations as an expansion in vanishing r can be found in [41]. Now let us apply the steps we discussed by giving a concrete example.

Example: System of Differential Equations for the basis of one loop massive integrals

In this section, we will illustrate the method of differential equations we just presented. Consider the family of integrals $\mathrm{I}\,[\nu_1,\nu_2,\nu_3]$:

$$\mathrm{I}\,[\nu_1,\nu_2,\nu_3] = e^{\epsilon\,\gamma_E}\int\frac{d^dk}{i\pi^{d/2}}\,\frac{1}{[k^2-m^2]^{\nu_1}\left[(k+p_1)^2-m^2\right]^{\nu_2}\left[(k+p_1+p_2)^2-m^2\right]^{\nu_3}} \tag{3.123}$$

with $p_1^2 = p_2^2 = 0$ and $(p_1+p_2)^2 = s$. We choose the following basis of master integrals:

$$\mathbf{I} = (\mathrm{I}\,[1,1,1]\,,\mathrm{I}\,[1,0,1]\,,\mathrm{I}\,[1,0,0])^{\mathrm{T}} \tag{3.124}$$

We obtain the differential equations with respect to m^2:

$$\partial_{m^2}\,\mathrm{I}\,[1,1,1] = \mathrm{I}\,[2,1,1] + \mathrm{I}\,[1,2,1] + \mathrm{I}\,[1,1,2] \tag{3.125}$$

$$\partial_{m^2}\,\mathrm{I}\,[1,0,1] = \mathrm{I}\,[2,0,1] + \mathrm{I}\,[1,0,2] \tag{3.126}$$

$$\partial_{m^2}\,\mathrm{I}\,[1,0,0] = \mathrm{I}\,[2,0,0] \tag{3.127}$$

Upon IBP reduction, we find the following relations:

$$\begin{aligned}\mathrm{I}\,[2,1,1] = &\left(\frac{(-1+\epsilon)\left(-\epsilon\, s + m^2\,(-2+4\epsilon)\right)}{m^4\,\left(4m^2-s\right)\,s}\right)\mathrm{I}\,[1,0,0]\\ &+\left(\frac{2\,(-1+2\epsilon)}{\left(4m^2-s\right)s}\right)\mathrm{I}\,[1,0,1]\end{aligned} \tag{3.128}$$

$$\mathrm{I}\,[1,2,1] = \left(\frac{-1+3\epsilon-2\epsilon^2}{m^4\,s}\right)\mathrm{I}\,[1,0,0] + \left(\frac{1-2\epsilon}{m^2\,s}\right)\mathrm{I}\,[1,0,1] - \frac{\epsilon}{m^2}\,\mathrm{I}\,[1,1,1] \tag{3.129}$$

$$\mathrm{I}\,[1,1,2] = \left(\frac{(-1+\epsilon)\left(-s\,\epsilon + 2\,m^2\,(-1+2\epsilon)\right)}{m^4\,\left(4m^2-s\right)s}\right)\mathrm{I}\,[1,0,0] + \left(\frac{2\,(-1+2\epsilon)}{\left(4m^2-s\right)\,s}\right)\mathrm{I}\,[1,0,1] \tag{3.130}$$

$$\mathrm{I}\,[2,0,1] = \mathrm{I}\,[1,0,2] = \left(\frac{-1+\epsilon}{m^2\,\left(4m^2-s\right)}\right)\mathrm{I}\,[1,0,0] + \left(\frac{-1+2\epsilon}{\left(-4m^2+s\right)}\right)\mathrm{I}\,[1,0,1] \tag{3.131}$$

$$\mathrm{I}\,[2,0,0] = \left(\frac{1-\epsilon}{m^2}\right)\mathrm{I}\,[1,0,0] \tag{3.132}$$

Plugging these relations into Equations 3.125, we obtain:

$$\partial_{m^2}\,\mathrm{I}\,[1,1,1] = \frac{-\epsilon}{m^2}\,\mathrm{I}\,[1,1,1] + \left(\frac{(-1+2\epsilon)}{m^2\left(4m^2-s\right)}\right)\mathrm{I}\,[1,0,1] - \left(\frac{(-1+\epsilon)}{m^4\,\left(4m^2-s\right)}\right)\mathrm{I}\,[1,0,0] \tag{3.133}$$

$$\partial_{m^2}\,\mathrm{I}\,[1,0,1] = \left(\frac{2\,(-1+2\epsilon)}{-4m^2+s}\right)\mathrm{I}\,[1,0,1] + \left(\frac{2\,(-1+\epsilon)}{m^2\left(4m^2-s\right)}\right)\mathrm{I}\,[1,0,0] \tag{3.134}$$

$$\partial_{m^2}\,\mathrm{I}\,[1,0,0] = \left(\frac{1-\epsilon}{m^2}\right)\mathrm{I}\,[1,0,0] \tag{3.135}$$

Now we can write these equations in a matrix form. Using the dimensionless parameter:

$$r = \frac{m^2}{s} \tag{3.136}$$

we obtain:

$$\partial_r \mathbf{I} = \mathrm{A}\,(r,\epsilon)\ \mathbf{I} \tag{3.137}$$

with

$$\mathrm{A}\,(r,\epsilon) = \begin{pmatrix} -\frac{\epsilon}{r} & \frac{2\epsilon-1}{r\,(4r-1)} & -\frac{\epsilon-1}{r^2\,(4r-1)} \\ 0 & -\frac{2\,(2\epsilon-1)}{4r-1} & \frac{2\,(\epsilon-1)}{r\,(4r-1)} \\ 0 & 0 & -\frac{\epsilon-1}{r} \end{pmatrix} \tag{3.138}$$

where we set $s = 1$. We see that the matrix $\mathrm{A}\,(r,\epsilon)$ contains a pole of $1/r^2$ at $r = 0$ which we can remove by applying the transformation $\mathrm{T}_{\mathrm{rank}}$:

$$\mathrm{T}_{\mathrm{rank}}(r) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{pmatrix} \tag{3.139}$$

The new matrix is then determined by using Equation 3.101. As an expansion in r, it reads:

$$\tilde{\mathrm{A}}(r,\epsilon) = \frac{\tilde{\mathrm{R}}\,(\epsilon)}{r} + \tilde{\mathrm{A}}_0\,(\epsilon) + \ldots + \mathcal{O}(r) \tag{3.140}$$

with

$$\tilde{\mathrm{R}}\,(\epsilon) = \begin{pmatrix} -\epsilon & 1-2\epsilon & \epsilon-1 \\ 0 & 0 & 0 \\ 0 & 0 & -\epsilon \end{pmatrix}, \quad \tilde{\mathrm{A}}_0\,(\epsilon) = \begin{pmatrix} 0 & -4\,(2\epsilon-1) & 4\,(\epsilon-1) \\ 0 & 2\,(2\epsilon-1) & -2\,(\epsilon-1) \\ 0 & 0 & 0 \end{pmatrix} \tag{3.141}$$

The eigenvalues of the matrix $\tilde{\mathrm{R}}\,(\epsilon)$ is already proportional to ϵ, therefore we do not need to perform a second transformation, $\mathrm{T}_{\mathrm{norm}}$, in this simple example. However, we still would like to put the matrix into Jordan Normal form. The Jordan decomposition of this matrix is:

$$\mathrm{S}[\tilde{\mathrm{R}}] = \begin{pmatrix} \frac{1-2\epsilon}{\epsilon} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\epsilon-1} \end{pmatrix}, \quad \mathrm{J}[\tilde{\mathrm{R}}] = \mathrm{S}[\tilde{\mathrm{R}}]^{-1}\tilde{\mathrm{R}}\ \mathrm{S}[\tilde{\mathrm{R}}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\epsilon & 1 \\ 0 & 0 & -\epsilon \end{pmatrix}, \tag{3.142}$$

Now we can write the solution to the system of differential equations we have using Equations 3.110 and 3.111:

$$\mathbf{I}\,(r,\epsilon) = \mathrm{T_{tot}}\, r^{\tilde{\mathrm{R}}(\epsilon)} \left(\sum_{n=0}^{\infty} \mathrm{D}_n\,(r,\epsilon)\right) \mathbf{BC}\,(\epsilon) \tag{3.143}$$

$$= \mathrm{T_{tot}}\, r^{\tilde{\mathrm{R}}(\epsilon)} \left(\mathbb{1} + \mathrm{D}_1\,(r,\epsilon) + \mathcal{O}(r^2)\right) \mathbf{BC}\,(\epsilon) \tag{3.144}$$

For the term $r^{\tilde{\mathrm{R}}(\epsilon)}$ we obtain

$$\begin{aligned} r^{\tilde{\mathrm{R}}(\epsilon)} &= \mathrm{S}[\tilde{\mathrm{R}}]\, r^{\mathrm{J}[\tilde{\mathrm{R}}]}\, \mathrm{S}[\tilde{\mathrm{R}}]^{-1} \\ &= \begin{pmatrix} r^{-\epsilon} & \left(1 - r^{-\epsilon}\right)\left(\frac{1}{\epsilon} - 2\right) & r^{-\epsilon}\,(\epsilon - 1)\log(r) \\ 0 & 1 & 0 \\ 0 & 0 & r^{-\epsilon} \end{pmatrix} \end{aligned} \tag{3.145}$$

where we evaluated $r^{\mathrm{J}[\tilde{\mathrm{R}}]} = \exp\{\mathrm{J}[\tilde{\mathrm{R}}]\, \log(r)\}$ as a matrix exponential.

The first order in the Dyson series is:

$$\mathrm{D}_1\,(r,\epsilon) = \int_0^r dr_1\, r_1^{-\tilde{\mathrm{R}}(\epsilon)}\, \mathrm{A}_0\,(r,\epsilon)\; r_1^{\tilde{\mathrm{R}}(\epsilon)} \tag{3.146}$$

we find:

$$\mathrm{D}_1\,(r,\epsilon) = \begin{pmatrix} 0 & -\dfrac{2\,r\;(2\epsilon-1)\left(r^{\epsilon} + 2\epsilon^2 + \epsilon - 1\right)}{\epsilon\,(\epsilon+1)} & \dfrac{2\,r\;\left((1-2\epsilon)r^{-\epsilon} + \epsilon - 1\right)}{\epsilon} \\ 0 & 2\,r\;(2\epsilon - 1) & 2\,r^{1-\epsilon} \\ 0 & 0 & 0 \end{pmatrix} \tag{3.147}$$

Now we can plug in the expressions for $\mathrm{T_{tot}}$, $r^{\tilde{\mathrm{R}}(\epsilon)}$ and $\mathrm{D}_1\,(r,\epsilon)$ in Equation 3.144. Defining the vector of boundary conditions as:

$$\mathbf{BC} = (\mathrm{BC}_1, \mathrm{BC}_2, \mathrm{BC}_3)^{\mathrm{T}} \tag{3.148}$$

We obtain the solutions as:

$$\begin{aligned} \mathrm{I}\,[1,1,1] = {} & r^{-\epsilon}\left(\mathrm{BC}_1 + \mathrm{BC}_2\left(2 - \frac{1}{\epsilon}\right) + \mathrm{BC}_3\,((\epsilon-1)\log(r)) + \mathcal{O}\,(r)\right) \\ & + \mathrm{BC}_2\left(-2 + \frac{1}{\epsilon} + \mathcal{O}\,(r)\right) \end{aligned} \tag{3.149}$$

$$\mathrm{I}\,[1,0,1] = r^{1-\epsilon}\left(2\,\mathrm{BC}_3 + \mathcal{O}\,(r)\right) + \left(\mathrm{BC}_2 + \mathcal{O}\,(r)\right) \tag{3.150}$$

$$\mathrm{I}\,[1,0,0] = r^{1-\epsilon}\Big(\mathrm{BC}_3 + \mathcal{O}\,(r)\Big) \tag{3.151}$$

Now we can use the results of certain regions we computed for $\mathrm{I}\,[1,1,1]$, $\mathrm{I}\,[1,0,1]$ and $\mathrm{I}\,[1,0,0]$ to determine the boundary conditions, which are given in Equations 3.90, 3.92 and 3.93 respectively. The first observation we make is that the boundary condition BC_3 can be determined directly from the $r^{1-\epsilon}$ scaling of the tadpole integral $\mathrm{I}\,[1,0,0]$. Matching the right hand sides of Equations 3.93 and 3.151, we obtain:

$$\begin{aligned}\mathrm{BC}_3 &= e^{\epsilon\,\gamma_E}\Big(-\Gamma\,(-1+\epsilon)\Big) &(3.152)\\ &= \frac{1}{\epsilon} + 1 + \left(1 + \frac{\pi^2}{12}\right)\epsilon + \mathcal{O}\left(\epsilon^2\right) &(3.153)\end{aligned}$$

where in the second line we expand in ϵ around zero. The boundary condition BC_2, on the other hand, can be determined from r^0 piece of the bubble integral $\mathrm{I}\,[1,0,1]$.

$$\begin{aligned}\mathrm{BC}_2 =&\ e^{-i\pi\,\epsilon}\,e^{\epsilon\,\gamma_E}\,\frac{\Gamma\,(1-\epsilon)^2\,\Gamma\,(\epsilon)}{\Gamma\,(2-2\epsilon)} &(3.154)\\ =&\ \frac{1}{\epsilon} + 2 - i\,\pi + \frac{1}{12}\left(48 - 24\,i\,\pi - 7\pi^2\right)\epsilon + \mathcal{O}\left(\epsilon^2\right) &(3.155)\end{aligned}$$

We can then take these results and plug them into Equation 3.149 to determine BC_1 from the $r^{-\epsilon}$ scaling of the triangle master $\mathrm{I}\,[1,1,1]$. This corresponds to $m^{-2\epsilon}$ piece in Equation 3.90. Let us write the $m^{-2\epsilon}$ scaling of this integral explicitly:

$$\mathrm{Tri}_{2\epsilon} = e^{\epsilon\,\gamma_E}\left(m^{-2\epsilon}\,(-s)^{-1}\,\Gamma\,(\epsilon)\left(-\gamma_E + \log\left(-\frac{m^2}{s}\right) - \psi^0\,(\epsilon)\right) + \mathcal{O}(m^2)\right) \tag{3.156}$$

$$= e^{\epsilon\,\gamma_E}\Big(-r^{-\epsilon}\,\Gamma\,(\epsilon)\,\Big(-\gamma_E + i\pi + \log\,(r) - \psi^0\,(\epsilon)\Big) + \mathcal{O}(r)\Big) \tag{3.157}$$

where in the second line we performed the analytic continuation. Combining everything we obtain:

$$\mathrm{BC}_1 = -\frac{2\,i\,\pi}{\epsilon} - \frac{\pi^2}{2} + \left(\frac{1}{6}\,i\,\pi^3\,\epsilon - 3\zeta_3\right)\epsilon + \mathcal{O}\left(\epsilon^2\right) \tag{3.158}$$

where we keep the imaginary pieces for completeness. It should be noted that the results we found for boundary conditions can also be cross-checked from the regions of the integrals apart from the ones we considered. For example, we can look at the $r^{-\epsilon}\,\log(r)$ piece of the massive triangle integral $\mathrm{I}\,[1,1,1]$ in Equation 3.157 and match it with the right hand side of Equation 3.149. We obtain:

$$\mathrm{BC}_3 = -\frac{\Gamma(\epsilon)}{\epsilon - 1} \tag{3.159}$$

which is the same as in Equation 3.152 when the relation $\Gamma(\epsilon) = (-1+\epsilon)\,\Gamma(-1+\epsilon)$ is employed. Similarly, BC_2 can be cross-checked by matching the m^0 scaling of the massive triangle integral in Equation 3.90 to the right hand side of Equation 3.149. The m^0 scaling in Equation 3.90 is:

$$\mathrm{Tri}_0 = e^{\epsilon\,\gamma_E}\Big(-\frac{(-s)^{-1-\epsilon}\Gamma(-\epsilon)^2\,\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} + \mathcal{O}(m^2)\Big) \tag{3.160}$$

$$= e^{\epsilon\,\gamma_E}\Big(e^{-i\pi\epsilon}\frac{(1-2\epsilon)\,\Gamma(1-\epsilon)^2\,\Gamma(\epsilon)}{\epsilon\,\Gamma(2-2\epsilon)} + \mathcal{O}(r)\Big) \tag{3.161}$$

where in the second line we rewrote the gamma functions using again the identity $\Gamma(n+1) = n\,\Gamma(n)$. The r^0 piece on the right hand side of Equation 3.149 on the other hand is:

$$\mathrm{BC}_2\left(-2 + \frac{1}{\epsilon} + \mathcal{O}(r)\right) \tag{3.162}$$

Matching these two equations, 3.161 and 3.162, we obtain the correct result for BC_2.

3.4 Feynman Integrals with Multiple Scales

In this thesis we deal with massive Feynman integrals: i.e. integrals that are a result of Feynman diagrams with massive bottom quarks running in the loop. We would like to have an analytic result of the cross section that is expanded in the small bottom quark mass, m_b. Different methods exist for expanding Feynman integrals in a small parameter. One of them is the method of *Expansion by Regions* [100, 101]. Here we will very briefly introduce expansion by regions.

Expansion by regions relies on the fact that one cannot simply expand a Feynman integral in a Taylor series with respect to a small parameter, since loosely speaking, the loop momenta is undetermined and it is not clear if it is not larger or smaller than the expansion parameter itself. Therefore in order to see the true asymptotic behaviour of the integral in this small parameter, the integral should be split into different regions corresponding to different behaviour of the loop momenta.

In practical calculations, how to apply the method of expansion by regions to a Feynman integral can be summarised as [101]:

- Divide the Feynman integral into regions with different boundaries of loop momenta; then expand the integrands in each region in Taylor series with respect to parameters that are small in the corresponding region.
- Change the boundaries of the loop momenta of each region to the whole integration domain; and perform the integrations within this domain.

In order to illustrate expansion by regions consider the massive triangle integral with massless legs incoming, where the bottom quarks are running in the loop:

$$\mathrm{Tri}(s, m_b^2) = e^{\epsilon\,\gamma_E} \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_1\, D_2\, D_3} \tag{3.163}$$

$$\text{with} \quad D_1 = k^2 - m_b^2 \tag{3.164}$$

$$D_2 = (k - p_1)^2 - m_b^2 \tag{3.165}$$

$$D_3 = (k + p_2)^2 - m_b^2 \tag{3.166}$$

where k is the loop momenta, p_1, p_2 are the external momenta with $(p_1 + p_2)^2 = s$, and m_b is the bottom quark mass. We consider the case where $m_b \ll s$, then we can have the following regions for this integral:

- Region 1 (hard region): $k^2 \sim p^2$. In this case the small parameter is m^2/k^2, and we can expand all the denominators around this parameter.
- Region 2 (soft/collinear region): k^2, $2k \cdot p_2 \sim m_b^2$, then we can expand D_2 around $m_b^2/(2k \cdot p_1)$.
- Region 3 (soft/collinear region): k^2, $2k \cdot p_1 \sim m_b^2$. We can expand D_3 around $m_b^2/(2k \cdot p_2)$.

These integrals can be computed separately and added together to obtain an approximate result for the triangle integral. Instead of expanding the propagators directly in the small parameters, let us express the original integral in Feynman parameters and determine the limits that the Feynman parameters can take according to each region, which amounts to a rescaling. We would like to regulate two of the propagators by adding a small exponent, $a \to 0$, which will become clear in a moment:

$$\mathrm{Tri}(s, m_b^2) = e^{\epsilon\,\gamma_E} \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_1^{1+a}\, D_2\, D_3^{1-a}} \tag{3.167}$$

The Feynman parameters representation for this integral is:

$$\begin{aligned}\mathrm{Tri}(s, m_b^2) = e^{\epsilon\,\gamma_E}\,\frac{-\Gamma\left(1+\epsilon\right)}{\Gamma\left(1-a\right)\Gamma\left(1+a\right)}\int dx_1\,dx_2\,dx_3\,\delta\left(1-x_1-x_2-x_3\right)x_1^a\,x_3^{-a}\\ \times\left(x_1+x_2+x_3\right)^{-1+2\epsilon}\left(-sx_1x_3+m_b^2\left(x_1+x_2+x_3\right)^2\right)^{-1-\epsilon}\end{aligned} \tag{3.168}$$

In this case again where $m_b \ll s$, we can have the following set of rescalings of the Feynman parameters:

$$\begin{aligned}&\text{Region 1}: \quad x_1 \to x_1, \quad x_2 \to x_2, \quad x_3 \to x_3\\ &\text{Region 2}: \quad x_1 \to x_1, \quad x_2 \to x_2, \quad x_3 \to m_b^2 x_3\\ &\text{Region 3}: \quad x_1 \to x_1, \quad x_2 \to m_b^2 x_2, \quad x_3 \to x_3\end{aligned}$$

With these rescalings, we obtain the following Feynman parametrizations up to $\mathcal{O}(m_b)$ for these three regions:

$$\begin{aligned}\mathrm{Tri}_{\mathrm{Region1}} =&(-s)^{-1-\epsilon}\mathcal{C}\int_0^1 dx_1\,dx_2\,dx_3\,\delta\left(1-x_1-x_2-x_3\right)x_1^a x_3^{-a}\\ &\times\left(x_1+x_2+x_3\right)^{-1+2\epsilon}\left(x_2x_3\right)^{-1-\epsilon} &(3.169)\\ \mathrm{Tri}_{\mathrm{Region2}} =&m_b^{-2\epsilon-2a}\mathcal{C}\int_0^1 dx_1\,dx_2\,dx_3\,\delta\left(1-x_1-x_2\right)x_1^a x_3^{-a}\left(x_1+x_2\right)^{-1+2\epsilon}\\ &\times\left(x_1^2-s\,x_2x_3+2x_1x_2+x_2^2\right)^{-1-\epsilon} &(3.170)\\ \mathrm{Tri}_{\mathrm{Region3}} =&m_b^{-2\epsilon}\mathcal{C}\int_0^1 dx_1\,dx_2\,dx_3\,\delta\left(1-x_1-x_3\right)x_1^a x_3^{-a}\left(x_1+x_3\right)^{-1+2\epsilon}\\ &\times\left(x_1^2+2x_1x_3+x_3^2-s\,x_2x_3\right)^{-1-\epsilon} &(3.171)\end{aligned}$$

where for convenience we denoted the common pre-factor as $\mathcal{C}$:

$$\mathcal{C} = -e^{\epsilon\,\gamma_E}\,\frac{\Gamma\left(1+\epsilon\right)}{\Gamma\left(1-a\right)\Gamma\left(1+a\right)} \tag{3.172}$$

As seen from the expressions above, for Region 2 and Region 3 the overall scaling is $m_b^{-2\epsilon}$, whereas a scaling of m_b^n with n being an integer corresponds to Region 1.

The calculation of these regions are straightforward using the tools introduced in Section 3.3. We get:

$$\mathrm{Tri}_{\mathrm{Region1}} = -(-s)^{-1-\epsilon}\,\frac{e^{\epsilon\,\gamma_E}\,\Gamma\left(-\epsilon\right)^2\Gamma\left(1+\epsilon\right)}{\Gamma\left(1-2\epsilon\right)} + \mathcal{O}(a) \tag{3.173}$$

$$\mathrm{Tri}_{\mathrm{Region2}} = -m_b^{-2\epsilon-2a}\,(-s)^{-1+a}\,\frac{e^{\epsilon\,\gamma_E}\,\Gamma\left(a\right)\Gamma\left(a+\epsilon\right)}{\Gamma\left(1+2a\right)} \tag{3.174}$$

$$\mathrm{Tri}_{\mathrm{Region3}} = m_b^{-2\epsilon}\,e^{\epsilon\,\gamma_E}\,\frac{\Gamma\left(-a\right)\Gamma\left(\epsilon\right)}{(-s)\Gamma\left(1-a\right)} \tag{3.175}$$

In this specific case, by splitting the original massive triangle integral into regions, spurious singularities are created, which are not regulated by dimensional regularization. Therefore we introduced a regularization parameter. The last two regions, when evaluated, have divergences which are now parameterized with a and will appear as $1/a$ poles when expanded. As expected, when these two regions are expanded in a around zero, these singularities cancel with each other. Up to $\mathcal{O}(a)$ we obtain:

$$\mathrm{Tri}_{\mathrm{Region1}} = -(-s)^{-1-\epsilon}\,\frac{e^{\epsilon\,\gamma_E}\,\Gamma\left(-\epsilon\right)^2\Gamma\left(1+\epsilon\right)}{\Gamma\left(1-2\epsilon\right)} \tag{3.176}$$

$$\begin{aligned}\mathrm{Tri}_{\mathrm{Region2}} + \mathrm{Tri}_{\mathrm{Region3}} &= e^{\epsilon\,\gamma_E}\,m_b^{-2\epsilon}\,(-s)^{-1}\,\Gamma\left(\epsilon\right)\\ &\quad\times\left(-\gamma_E + \log(m_b^2) - \log(-s) - \psi^{(0)}(\epsilon)\right)\end{aligned} \tag{3.177}$$

where for completeness we also expressed the result from the first region again. Looking at these expressions, we see that these results are precisely the same, as expected, with the results in Equation 3.87 and Equation 3.89 that we obtained for the massive triangle calculation with the Mellin Barnes method, giving the final result:

$$\begin{aligned}\mathrm{Tri} = e^{\epsilon\,\gamma_E}\Bigg(&-\frac{(-s)^{-1-\epsilon}\,\Gamma\left(-\epsilon\right)^2\Gamma\left(1+\epsilon\right)}{\Gamma\left(1-2\epsilon\right)}\\ &+ m_b^{-2\epsilon}\,(-s)^{-1}\,\Gamma\left(\epsilon\right)\left(-\gamma_E + \log\left(-\frac{m_b^2}{s}\right) - \psi^{(0)}(\epsilon)\right)\\ &+ \mathcal{O}\left(m_b^2\right)\Bigg)\end{aligned} \tag{3.178}$$

The integral that we just discussed is a one loop integral with three external legs, however as the loop and propagator number increases, many possibilities for different regions arise. Therefore determining regions become highly non-trivial, such that doing this task by hand can be very tedious. There is a MATHEMATICA program called ASY [110, 111], which automatically determines the regions. We use this program to extract the regions for the integrals that appear in the double-virtual contribution for top-bottom interference.

Another way to have an asymptotic expansion of Feynman integrals in small parameters is to directly use the Mellin Barnes Representation method, as we saw in Section 3.3.2. With this approach, the regions do not have to be determined from the beginning, but are revealed naturally as a consequence of picking up the residues using Cauchy's theorem for a given integral in Mellin Barnes representation. In this thesis we prefer to use this method for the master integrals appearing in the real-virtual contribution. This will be the topic of Section 6.2.

Chapter 4

Inclusive Cross Section in QCD

Because the nature of QCD relies on *asymptotic freedom*, perturbative calculations on the hadrons directly is not possible since at this scale the strong coupling is not small enough. However this feature also allows us to have a model, the *Parton Model*, which we can use to make phenomenological predictions for hadron colliders such as Tevatron or LHC. The Parton Model states that for interactions where the momentum transfer Q is much larger than the confinement scale Λ_{QCD}, the partons can be treated as free and the *hard scattering* that occurs between the partons is assumed to be independent of the hadronization process that happens at a later time. We call this, *factorization*, and what we treat perturbatively is the hard scattering.

According to factorization, the hadronic cross section is given as the convolution of partonic cross section $\hat{\sigma}$ over the parton distribution functions $f_i(x_1)$ and $f_j(x_2)$:

$$\sigma_{p\,p\to X} = \sum_{i,j} \int_0^1 dx_1\, dx_2\, f_i(x_1, \mu_F^2)\, f_j(x_2, \mu_F^2)\, \hat{\sigma}_{i\,j\to X} \tag{4.1}$$

where we sum over all types of partons i and j. $\hat{\sigma}_{i\,j\to X}$ is the partonic cross section that describes the scattering of two partons which carry fractions of initial momenta: $p_i = x_1\, P_1$ and $p_j = x_2\, P_2$, where P_1 and P_2 are the momenta of the initial state hadrons. $f_i(x_1, \mu_F^2)$ and $f_j(x_2, \mu_F^2)$ are the parton distribution functions (PDF's) that stand for the probability to find a parton i or j with the corresponding fraction of momenta inside a proton, so they are directly related to the structure of a proton, and independent of the process of interest, i.e. they are universal. Since they are

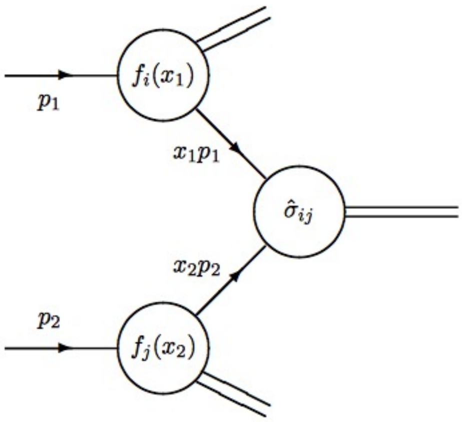


Figure 4.1: Factorization in QCD

universal, the PDF's can be extracted from experiments. Additionally, they do not belong to the hard scattering therefore, we cannot compute them perturbatively. μ_F^2 is the factorization scale, which is an unphysical scale that we introduce to distinguish between the hard scattering and the non-perturbative effects. The hadronic cross section depends on the factorization scale μ_F^2, and this dependence can be lessened by calculating more and more terms in the perturbative series of the partonic cross section. An analysis of scale variation of the hadronic cross section can indicate if the reached order in perturbative series is a good approximation.

As mentioned earlier, the partonic cross section, which is a quantity describing the hard scattering, is what we can calculate in perturbation theory. We can write it as an expansion in the strong coupling α_s as:

$$\hat{\sigma} = \alpha_s(\mu^2)\,\sigma^{(0)} + \alpha_s^2(\mu^2)\,\sigma^{(1)} + \cdots \tag{4.2}$$

One can then start calculating term by term in the series. The first term $\sigma^{(0)}$, i.e. the leading order (LO), is usually not sufficient; and in order to make a reliable phenomenological prediction, we would like to increase the accuracy by calculating as many terms as possible in the series. However, as the α_s order increases, the complexity increases immensely, and calculating higher orders becomes a tedious task. As mentioned in the Introduction, in recent years there is enormous progress made in precision physics.

From now on we will drop the hat, and denote the *inclusive cross section* as σ. Inclusive cross section means we take into account any potential configuration of the momenta of the final state particlesregardless of the constraints due to the experimental set-up, i.e. we will integrate over the whole phase space:

$$\sigma\left(\{p_i, m_i\}\right) = \frac{1}{\text{flux}} \int d\Phi_n\left(\{p_i, m_i\}\right) \left|\overline{\mathcal{M}}\left(\{p_i, m_i\}\right)\right|^2 \tag{4.3}$$

where $\overline{\mathcal{M}}\left(\{p_i, m_i\}\right)$ is the matrix element squared, that is summed over the final state spins and colors and averaged over initial state spins and colors. $\{p_i, m_i\}$ is the list of momenta and masses of the initial and final state particles respectively.

To include higher order corrections, we should add both real and virtual contributions together in order to obtain a result at a certain order. For instance for the process $g\,g \to H$, up to order α_s^3 (NLO), dropping the flux factor and summing/averaging pre-factors for convenience, we have:

$$\sigma_{g\,g\to H} = \int d\Phi_2 \left|\mathcal{M}^B\right|^2 + \int d\Phi_2\, 2\Re\left\{\left(\mathcal{M}^B\right)^* \mathcal{M}^V\right\} + \int d\Phi_3 \left|\mathcal{M}^R\right|^2 \tag{4.4}$$

where $\mathcal{M}^B$ is the Born matrix element, whereas $\mathcal{M}^V$ and $\mathcal{M}^R$ are the virtual and real contributions respectively.

4.1 Phase Space Measure

The d-dimensional phase space measure is:

$$d\Phi_n\left(\{p_i, m_i\}\right) = (2\pi)^d\, \delta^d\left(q_1 + q_2 - \sum_{i=1}^{n} p_i\right) \left(\prod_{i=1}^{n} \frac{d^d p_i}{(2\pi)^{d-1}} \delta^+\left(p_i^2 - m_i^2\right)\right) \tag{4.5}$$

where we used the notation $\delta^+\left(p_i^2 - m_i^2\right) = \delta\left(p_i^2 - m_i^2\right)\theta(p^0)$; and p_i are the momenta of the final state particles, whereas q_1 and q_2 are the momenta of the initial state particles. The d dimensional delta function ensures the momentum conservation.

In this thesis we are interested in the Higgs production via gluon fusion, so we need to consider processes $g\,g \to H$ and $g\,g \to H + X$, where X is a radiated massless parton produced along with the Higgs. Therefore we need to use the phase space measures for one and two particles, both of which we shall derive now starting from Equation 4.5. The matrix element of the top-bottom interference for the process $g\,g \to H$ at LO and NLO is independent of phase space variables, therefore we can use the phase space volume:

$$\begin{aligned}\Phi_{2\to 1}\left(p_1, p_2, p_H; m_H\right) =& \int d\Phi_{2\to 1} = (2\pi) \int d^d p_H\, \delta^+\left(p_H^2 - m_H^2\right) \delta^d\left(p_H - p_1 - p_2\right) \\ =& \frac{2\pi}{m_H^2}\, \delta\left(1 - z\right) \end{aligned} \tag{4.6}$$

where p_1 and p_2 are the momenta of the incoming particles and:

$$z = \frac{m_H^2}{s}, \quad m_H^2 = p_H^2 \tag{4.7}$$

The $2 \to 2$ particle phase space measure on the other hand is:

$$\begin{aligned}d\Phi_{2\to 2}\left(p_1, p_2, p_3, p_H; m_H\right) =& \frac{d^d p_H}{(2\pi)^{d-1}} \frac{d^d p_3}{(2\pi)^{d-1}}\, (2\pi)^d\, \delta^+\left(p_H^2 - m_H^2\right)\, \delta^+\left(p_3^2\right) \\ & \times \delta^d\left(p_H + p_3 - p_1 - p_2\right)\end{aligned} \tag{4.8}$$

where p_3 is the outgoing massless particle produced with the Higgs. We can parameterize the phase space by defining the kinematical invariants as the following:

$$s_{ij} = 2 p_i \cdot p_j \tag{4.9}$$

$$s_{12} = s \tag{4.10}$$

$$s_{13} = s\,(1 - z)\lambda = s\,\bar{z}\lambda \tag{4.11}$$

$$s_{23} = s\,(1 - z)(1 - \lambda) = s\,\bar{z}\bar{\lambda} \tag{4.12}$$

$$z = \frac{m_H^2}{s} \tag{4.13}$$

Using these definitions we obtain:

$$d\Phi_2 = d\lambda \left(\lambda\bar{\lambda}\right)^{-\epsilon} \frac{(4\pi)^{-1+\epsilon}\, s^{-\epsilon}\, \bar{z}^{1-2\epsilon}}{2\Gamma\left(1 - \epsilon\right)} \tag{4.14}$$

4.2 Reverse Unitarity

Unitarity of the S-matrix results in the optical theorem, which relates the forward scattering amplitudes to phase space integrals. It allows the phase space integrals to be extracted from the discontinuities of the loop integrals [112–116]. This method has been used widely to compute cross sections [117–119].

With the *Reverse Unitarity* method [120–124] one does the opposite and write the phase space integrals at hand as unitarity cuts of the loop integrals [125]. This amounts to replacing the on-shell delta functions corresponding to phase space particles with discontinuities of the loop propagators:

$$2\pi i\delta^{+}(p^2 - m^2) = \mathrm{Disc}\,\frac{1}{p^2 - m^2 + i0}. \tag{4.15}$$

In the reverse unitarity approach the phase-space integrals are subject to the same IBP identities as their loop integrals. Therefore the phase-space integrals can be reduced to master integrals, and differential equations can be obtained in a similar manner. The exception is that integrals that have vanishing or negative power for any of the cut propagators must vanish. The method has been successfully applied to a variety of cut integrals [121–124, 126].

4.3 Plus Prescription

For the inclusive cross section, some of the singularities from real contribution and virtual contribution should cancel with each other; whereas the initial state IR divergences should be absorbed to PDF's by cancelling these poles with the so-called PDF counter-terms. However the integration of the partonic cross section over the PDF's is done numerically and if one wants to obtain a stable numerical result, it is more convenient to be able to isolate the poles and cancel them analytically before. *Plus prescription* is a subtraction method to extract the poles in ϵ. It amounts to rewriting an integral that have a singularity in the following way to make the pole structure manifest:

$$\int_0^1 dx\, x^{-1+a\epsilon} f(x) = f(0)\int_0^1 dx\, x^{-1+a\epsilon} + \int_0^1 dx\, x^{-1+a\epsilon}\left(f(x) - f(0)\right) \tag{4.16}$$

$$=\frac{f(0)}{a\epsilon}+\sum_{n=0}^{\infty}\left(\frac{a\epsilon}{n!}\right)^n\int_0^1 dx\,\log^n(x)\left(\frac{f(x)-f(0)}{x}\right) \tag{4.17}$$

where $f(x)$ is a function that is finite at $x=1$. In the second line we integrated the first term that resulted in a $1/\epsilon$ pole, and in the second term we expressed the $x^{-1+a\epsilon}$ as a Taylor expansion in ϵ. We can now remove the test function $f(x)$ and express $x^{-1+a\epsilon}$ as a distribution:

$$x^{-1+a\epsilon}=\frac{\delta(x)}{a\epsilon}+\sum_{n=0}^{\infty}\left(\frac{a\epsilon}{n!}\right)^n\mathcal{D}_n(x) \tag{4.18}$$

where $\mathcal{D}_n(x)$ is the plus-distribution:

$$\mathcal{D}_n(x)=\left[\frac{\log^n(x)}{x}\right]_+ \tag{4.19}$$

which acts on the functions as:

$$\int_0^1 dx\,\left[\frac{g(x)}{x}\right]_+ f(x)=\int_0^1 dx\,g(x)\left(\frac{f(x)-f(0)}{x}\right) \tag{4.20}$$

We will use this trick to isolate the soft singularities that appear in the real-virtual contribution of top-bottom interference cross section, which we will describe in Chapter 5.

4.4 Matrix Elements

The direct strategy to compute the inclusive cross section starts with the generation of Feynman diagrams and applying the Feynman rules to them. The result is an expression of scalar products of momenta involved in the process with an integration over the loop momenta and phase space. Most of these integrals are related to one another by means of IBP and Lorentz identities, that we already mentioned in previous chapter. The resulting integrals that cannot be further expressed in terms of others are called *Master Integrals*. In this section we will explain how to obtain matrix elements in terms of master integrals. We will first start with topology mapping, then describe *tensor reduction*, and IBP reduction. We will also give a concrete example by looking at the leading order for the production of Higgs boson via gluon fusion.

4.4.1 Tensor reduction

In calculating matrix elements involving external vector bosons the expressions before squaring the matrix element involves open Lorentz indices which should be contracted with the polarization vectors. In multi-loop amplitudes these open indices can appear in Feynman integrals as a Lorentz index of the loop momenta. However since the loop momenta is being integrated over, we know that the end result should not depend on loop momenta. Using this fact, in practical calculations it is simpler to re-express these terms in terms of the Lorentz vectors that the matrix element would depend on. This is called *tensor reduction* and can be done by contracting the matrix element with so-called *projectors*, or apply *Passarino-Veltman Reduction* [127]. In Passarino-Veltman reduction we propose an ansatz for the matrix element in terms of available Lorentz vectors for the particular process that we are interested in. As an example consider the process $g\,g \to h$ at leading order. The matrix element can be written as:

$$\mathcal{M}^{\mu\nu}_{ans} = A g^{\mu\nu} + B\, p_1^\mu p_1^\nu + C\, p_1^\mu p_2^\nu + D\, p_2^\mu p_2^\nu + E\, p_1^\nu p_2^\mu \tag{4.21}$$

where p_1 and p_2 are the momenta of the external gluons. We know that the matrix element should be symmetric in indices μ and ν, so this means the coefficients C and E must be equal to each other. Also, when contracted with the polarization vectors the terms involving the coefficients B and D will vanish because of the Ward identity for gluons despite the fact that these coefficients may not be zero:

$$\epsilon(p_1) \cdot p_1 = 0, \quad \epsilon(p_2) \cdot p_2 = 0 \tag{4.22}$$

The remaining coefficients A, C and E can be extracted by constructing three different scalar objects for both the ansatz and the matrix element that is already calculated. For the ansatz we have:

$$g_{\mu\nu}\mathcal{M}^{\mu\nu}_{ans} = Ad + C\,(p_1 \cdot p_2) + E\,(p_1 \cdot p_2) \tag{4.23}$$

$$p_1^\mu p_2^\nu\, \mathcal{M}^{\mu\nu}_{ans} = A\,(p_1 \cdot p_2) + E\,(p_1 \cdot p_2)^2 \tag{4.24}$$

$$p_1^\nu p_2^\mu\, \mathcal{M}^{\mu\nu}_{ans} = A\,(p_1 \cdot p_2) + C\,(p_1 \cdot p_2)^2 \tag{4.25}$$

where we used $g^{\mu\nu} g_{\mu\nu} = d$ signifying that we work in dimensional regularization. Repeating this also for the matrix element that is obtained by applying Feynman rules that we will label as $\mathcal{M}^{\mu\nu}$, we will have three equations with three unknowns which we can put in matrix form and solve the system of linear equations:

$$\begin{pmatrix} d & (p_1\cdot p_2) & (p_1\cdot p_2) \\ (p_1\cdot p_2) & 0 & (p_1\cdot p_2)^2 \\ (p_1\cdot p_2) & (p_1\cdot p_2)^2 & 0 \end{pmatrix} \begin{pmatrix} A \\ C \\ E \end{pmatrix} = \begin{pmatrix} g_{\mu\nu}\mathcal{M}^{\mu\nu} \\ p_1^\mu p_2^\nu\,\mathcal{M}^{\mu\nu} \\ p_1^\nu p_2^\mu\,\mathcal{M}^{\mu\nu} \end{pmatrix} \tag{4.26}$$

After determining the unknown coefficients the matrix element can be written in the form of Equation 4.21. Having already concluded that the coefficients C and E must be equal, we can also write the matrix element in a more compact form by noting that the coefficients A and E must also be related by Ward identities. Since gluons are massless and therefore only transverse we can write:

$$p_1^\mu p_2^\nu\,\mathcal{M}^{\mu\nu} = 0 \tag{4.27}$$

which leads to:

$$p_1^\mu p_2^\nu\,\mathcal{M}^{\mu\nu} = A\,(p_1\cdot p_2) + E\,(p_1\cdot p_2)^2 = 0 \tag{4.28}$$

$$E = -\frac{A}{(p_1\cdot p_2)} \tag{4.29}$$

Confirming that this is the case can also be a good check for the calculation of matrix element. Now we can write:

$$\mathcal{M}^{\mu\nu} = A\Big(g^{\mu\nu} - \frac{p_1^\nu p_2^\mu}{(p_1\cdot p_2)}\Big) \tag{4.30}$$

Apart from Passarino-Veltman reduction, there is another way of reducing the tensor structure to the form of Equation 4.30. Knowing that we can use the Ward identities for the process $g\,g \to h$, we can act on the matrix element, $\mathcal{M}^{\mu\nu}$, we calculated with a projector to extract the coefficient E:

$$P_{\mu\nu}\mathcal{M}^{\mu\nu} = A, \tag{4.31}$$

$$\text{with} \quad P_{\mu\nu} = \frac{1}{2-d}\Big(- g^{\mu\nu} + \frac{p_1^\mu p_2^\nu + p_1^\nu p_2^\mu}{p_1\cdot p_2}\Big) \tag{4.32}$$

For real radiation, i.e. $g\,g \to h\,g$ the tensor structure of the matrix element $\mathcal{M}^{\mu\nu\rho}$ is more complicated than the Equation 4.21 since now we have three Lorentz indices, which would also lead to a bigger system of linear equations. For this reason, in this thesis we calculate the real-virtual contribution by directly squaring the matrix elements, then we subsequently employ the reverse unitarity to compute the phase space integrals. For details see Chapter 6.

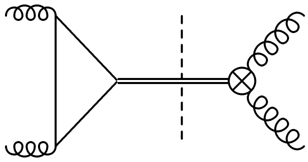


Figure 4.2: The leading order (LO) top-bottom interference contribution to the $gg \to H$ cross section

After performing the tensor reduction there remains the task of calculating Feynman integrals. As mentioned in Section 3.2.1, for multiloop calculations the integrals to calculate increase immensely as one goes from one order to the next in perturbation theory. However we can use the IBP relations among Feynman integrals to reduce them to fewer integrals, which are called the master integrals.

In the next section, we will give a small example about this chapter: calculation of the cross section for top-bottom interference for the process $g\,g \to H$ at LO.

4.5 Example: Cross Section for Top-Bottom Interference at LO

Consider for example the process of Higgs production via gluon fusion at leading order, i.e. $g\,g \to H$, which is loop induced since Higgs field does not couple to gluons directly. We will assume that only bottom quarks are running in the loop, and interfere this with the effective vertex for the same process where the top quark is integrated out. This will give us the LO contribution to the cross section for top-bottom interference, the figure of which is depicted in Figure 4.2.

Let us start with the amplitude. The matrix element for this process can be written generically as:

$$i\,\mathcal{M}(g\,g \to H) = \epsilon_\mu^{\lambda_1}(p_1)\epsilon_\nu^{\lambda_2}(p_2)\,C^{ab}\,\mathcal{M}^{\mu\nu} \tag{4.33}$$

where ϵ_μ and ϵ_ν are the polarization vectors and p_1 and p_2 are the momenta of the incoming gluons. λ_1 and λ_2 are the helicity indices of the gluons; $C^{ab} = \delta^{ab}/2$ is the color factor. We can contract the amplitude with a projector to eliminate the open Lorentz indices μ, ν. Recall from Equation 4.32 that, for this process the projector we use is:

$$P_{\mu\nu}\,\mathcal{M}^{\mu\nu} = A, \tag{4.34}$$

such that

$$\sum_{\lambda_1,\lambda_2} \epsilon_\mu^{\lambda_1}(p_1)\epsilon_\rho^{*\lambda_1}(p_1)\epsilon_\nu^{\lambda_2}(p_2)\epsilon_\sigma^{*\lambda_2}(p_2)\,P^{\mu\nu}\,P^{\rho\sigma} = \frac{1}{d-2} \tag{4.35}$$

$$\text{with} \quad \sum_{\lambda_1} \epsilon_\mu^{\lambda_1}(p_1)\epsilon_\rho^{*\lambda_1}(p_1) = -g_{\mu\rho} \tag{4.36}$$

The cross section for the interference can be written as:

$$\sigma_{\text{LO}} = \frac{1}{2s}\int d\Phi_2(p_1,p_2,p_H)\,2\Re\,\{\mathcal{M}^0_{\text{full}}\,\mathcal{M}^{0*}_{\text{eff}}\} \tag{4.37}$$

where $\mathcal{M}^0_{\text{full}}$ and $\mathcal{M}^0_{\text{eff}}$ denote the matrix elements in the full and effective theory respectively at leading order. Using these relations in Equation 4.35, we can write the matrix elements as:

$$\begin{aligned} 2\Re\,\{\mathcal{M}^0_{\text{full}}\,\mathcal{M}^{0*}_{\text{eff}}\} =& C^{ab}C_{ab}\sum_{\lambda_1,\lambda_2} \epsilon_\mu^{\lambda_1}(p_1)\epsilon_\rho^{*\lambda_1}(p_1)\epsilon_\nu^{\lambda_2}(p_2)\epsilon_\sigma^{*\lambda_2}(p_2)\,P^{\mu\nu}\,P^{\rho\sigma} \\ & \times\ 2\Re\,\{A_{\text{full}}\,A^*_{\text{eff}}\} && (4.38) \\ =& \frac{N_C^2-1}{4(d-2)}\,2\Re\,\{A_{\text{full}}\,A^*_{\text{eff}}\} && (4.39) \end{aligned}$$

Now we can write the matrix elements explicitly for both sides. The effective part can be expressed trivially using the gluon-gluon-Higgs vertex that we introduced in Section 2.2:

$$P_{\mu\nu}\,\mathcal{M}^{\mu\nu}_{\text{eff}} = A_{\text{eff}} \tag{4.40}$$

$$A_{\text{eff}} = \frac{is\,c_H}{2} \tag{4.41}$$

where c_H is the Wilson coefficient and v is the vacuum expectation value of the Higgs field, $s = (p_1+p_2)^2 = m_H^2$ is the center of mass energy.

The bottom quark part on the other hand can be expressed after applying the

Feynman rules as:

$$\mathcal{M}^{\mu\nu}_{\text{full}} = \int \frac{d^d k}{i\pi^{d/2}} \frac{\mathcal{N}^{\mu\nu}}{D_1 D_2 D_3} \tag{4.42}$$

where g_s is the strong coupling and m is the mass of the fermions running in the loop. The denominators are $D_1 = k^2 - m_b^2, D_2 = (k-p_1)^2 - m_b^2, D_3 = (k-p_1-p_2)^2 - m_b^2$. Acting with the projector we get the coefficient A_{full}:

$$P_{\mu\nu}\,\mathcal{M}^{\mu\nu}_{\text{full}} = A_{\text{full}} \tag{4.43}$$

The numerator $\mathcal{N}^{\mu\nu}$, on the other hand, is:

$$\mathcal{N}^{\mu\nu} = -\,g_s^2\,y_b\,\mathrm{Tr}\left[(\not{k}+m_b)\,\gamma^\mu((\not{k}-\not{p}_1)+m_b)\,\gamma^\nu((\not{k}-\not{p}_1-\not{p}_2)+m_b)\right] \tag{4.44}$$

$$\begin{aligned} &= -\,4g_s^2\,y_b\,m_b\Big(g^{\mu\nu}\,\left(2k\cdot p_1 - k^2 - p_1\cdot p_2 + m_b^2\right) \\ &\qquad + p_1^\mu\,(-2k^\nu + 2p_1^\nu + p_2^\nu) + k^\mu\,(4k^\nu - 2(2p_1^\nu + p_2^\nu)) + p_1^\nu p_2^\mu\Big) \end{aligned} \tag{4.45}$$

where y_b is the bottom quark Yukawa coupling, and g_s is the strong coupling. Then after acting with the projector, A_{full} becomes:

$$\begin{aligned} A_{\text{full}} =& \frac{8g_s^2 m_b^2}{v(2-d)} \int \frac{d^d k}{i\pi^{d/2} D_1 D_2 D_3} \\ &\times \left(2k\cdot p_1\left(-\frac{2k\cdot p_2}{p_1\cdot p_2} + d - 3\right) - (d-5)k^2 - (d-2)p_1\cdot p_2 + (d-1)m_b^2\right) \end{aligned} \tag{4.46}$$

where we plugged in $y_b = m_b/v$. Now we can rewrite the scalar products in terms of denominators:

$$k_1^2 = D_1 + m_b^2, \qquad k_1\cdot p_1 = \frac{D_1 - D_2}{2}, \tag{4.47}$$

$$k_1\cdot p_2 = \frac{D_2 - D_3 + s}{2}, \qquad p_1\cdot p_2 = \frac{s}{2} \tag{4.48}$$

Plugging these in, we obtain:

$$\begin{aligned} A_{\text{full}} =& \frac{4g_s^2 m_b^2}{v(2-d)}\Bigg(I\,[1,1,1]\left(8m_b^2 + s(2-d)\right) - 2I\,[1,0,1]\,(d-5) \\ &+ \frac{4}{s}\Big(I\,[0,1,0] - I\,[0,0,1] - I\,[1,0,0] + I\,[1,-1,1]\Big)\Bigg) \end{aligned} \tag{4.49}$$

which after the reduction, with full mass dependence, reads:

$$A_{\text{full}} = \frac{4g_s^2 m_b^2}{v(2-d)} \left(\mathrm{I}\,[1,1,1] \left(8m_b^2 - (d-2)s\right) - 2\,(d-4)\,\mathrm{I}\,[1,0,1]\right) \tag{4.50}$$

where we used the notation:

$$\mathrm{I}[\nu_1,\nu_2,\nu_3] = e^{\epsilon\gamma_E} \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3}} \tag{4.51}$$

such that $\mathrm{I}\,[1,1,1]$ and $\mathrm{I}\,[1,0,1]$ are massive triangle and bubble integrals. We also need to take into account the second diagram where the fermion charge flow is in the opposite direction. The contribution of this diagram is exactly the same, therefore we just multiply the result by two.

Now we can plug in the analytic expressions for the master integrals. We already calculated the triangle as an expansion in the bottom mass in Section 3.3.2, the result of which is given in Equation 3.90:

$$\begin{aligned}\mathrm{I}\,[1,1,1] = e^{\epsilon\gamma_E} \Bigg(&- \frac{(-s)^{-1-\epsilon}\,\Gamma\,(-\epsilon)^2\,\Gamma\,(1+\epsilon)}{\Gamma\,(1-2\epsilon)} \\ &+ m_b^{-2\epsilon}\,(-s)^{-1}\,\Gamma\,(\epsilon) \left(-\gamma_E + \log\left(-\frac{m_b^2}{s}\right) - \psi^{(0)}(\epsilon)\right) \\ &+ \mathcal{O}\left(m_b^2\right)\Bigg)\end{aligned} \tag{4.52}$$

$$\mathrm{I}\,[1,0,1] = e^{\epsilon\gamma_E} \left(\frac{(-s)^{-\epsilon}\Gamma\,(1-\epsilon)^2\,\Gamma\,(\epsilon)}{\Gamma\,(2-2\epsilon)} + \mathcal{O}\left(m_b^2\right)\right) \tag{4.53}$$

Plugging these expressions for the masters and expanding in bottom quark mass and ϵ we obtain:

$$\begin{aligned}2\Re\,\{\mathcal{M}^0_{\text{full}}\,\mathcal{M}^{0*}_{\text{eff}}\} =\tilde{\alpha}_s\,\mathcal{B} \Bigg\{ &-4 - 6\zeta_2 + L^2 \\ &+ \epsilon\left(-8 + 6\zeta_2 - 6\zeta_3 - 2\zeta_2 L - L^2 - \frac{2L^3}{3}\right) \\ &+ \epsilon^2\left(-16 + 14\zeta_2 + 6\zeta_3 + \frac{25\zeta_4}{2} + 2\,(\zeta_2+\zeta_3)\,L + \frac{3\zeta_2 L^2}{2} + \frac{2L^3}{3} + \frac{L^4}{4}\right)\end{aligned}$$

$$
\begin{aligned}
&+\epsilon^3\Bigg(-32+28\zeta_2+\frac{28\zeta_3}{3}-\frac{25\zeta_4}{2}+15\zeta_2\zeta_3-14\zeta_5-\left(2\zeta_3+\frac{9\zeta_4}{2}\right)L \\
&\qquad -\frac{1}{6}\left(9\zeta_2+8\zeta_3\right)L^2-\frac{2\zeta_2 L^3}{3}-\frac{L^4}{4}-\frac{L^5}{15}\Bigg) \\
&+\mathcal{O}\left(\epsilon^4\right)\Bigg\}+\mathcal{O}\left(m_b^4\right)
\end{aligned}
\tag{4.54}
$$

where

$$
\mathcal{B}=\frac{c_H\, m_b^2\,\left(N_C^2-1\right)}{v\,(-1+\epsilon)^2}, \tag{4.55}
$$

$$
\tilde{\alpha}_s=\left(\frac{s}{\mu}\right)^{-\epsilon} e^{-\epsilon\gamma_E}\,(4\pi)^{\epsilon}\,\frac{\alpha_s}{4\pi},\quad L=\log\left(\frac{m_b^2}{s}\right) \tag{4.56}
$$

Combining the result we found with the integration over phase space is trivial since two-body phase space factors out from the matrix element squared. We will write the full cross section result for LO in the next chapter for completeness.

Part II

Top-Bottom Interference for Higgs Production in Gluon Fusion at NLO in QCD

Chapter 5

The Cross Section at NLO

In this second part of the thesis we will describe how we analytically compute the cross section for top-bottom interference contribution to the Higgs production in gluon fusion at NLO in QCD. As already mentioned in the Introduction, by top-bottom interference we mean the process $g\,g \to H$ in full theory where the bottom quarks are running in the loop interfered with the same process with top quarks running in the loop. This is a two loop process with three different scales m_b^2, m_t^2 and m_H^2, which is highly difficult to compute. Here we make an expansion around $m_t \to \infty$ and $m_b \to 0$. We use heavy quark effective field theory and integrate out the top quark. The typical diagrams for this is given in Figure 5.1.

The full partonic cross section for the process $g\,g \to H$ can be written in two pieces; the first one being a result of Higgs production in the effective theory where the top quark is integrated out; and the second piece being a result of the interference between the process in the effective theory and in the full theory where only bottom quarks are running in the loop. Here we are only interested in the first term in the expansion of m_b^2, and we ignore the contributions coming from lighter quarks running in the loop:

$$\sigma_{ij\to RH} = \sigma^{\text{eff.}}_{ij\to RH} + \sigma^{\text{int.}}_{ij\to RH} + \mathcal{O}(y_b^2), \tag{5.1}$$

where 'int.' stands for interference and 'eff.' for effective theory, and y_b is the Yukawa coupling for the bottom quark. The contributions to the cross section $\sigma^{\text{eff.}}_{ij\to H}$ can be found in the literature up to α_s^5 [25]. The cross-section $\sigma^{\text{int.}}_{gg\to H}$ is the top-bottom interference and serves as to determine the lowest contribution up to order m_b^2, or

$y_b m_b$ at a certain α_s order; as opposed to the same process at the same α_s order where we work in the full theory with only bottom quarks running in the loops.

We calculate the cross section as an expansion in the bottom quark mass and in the dimensional regularization parameter ϵ, up to order ϵ. Our strategy to obtain the expansion is to use differential equations. Then we calculate the necessary regions for the master integrals, i.e. the *different scalings*, with Mellin Barnes Representation method to determine the boundary conditions. We will describe how we obtain the scalings in Section 6.2.

In this chapter we will give the results for the cross section and explain the different pieces contributing to it. The next two chapters will be dedicated to to these different pieces: real-virtual and double-virtual contributions. We will present the matrix elements, together with the master integrals needed in these chapters.

5.1 Set-up for the Calculation

As explained in the previous chapter, the total hadronic cross section for the Higgs production can be written as

$$\sigma^{\text{tot.}}_{p\,p\to H} = \sum_{i,j} \sum_{R} \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2\, f_i(x_1) f_j(x_2)\, \hat{\sigma}_{ij\to R\,H} \tag{5.2}$$

where f_i and f_j are the parton distribution functions for partons of type i and $\hat{\sigma}_{ij\to R\,H}$ is the partonic cross section for the process $i(p_1) + j(p_2) \to R(p_3, \ldots) + H(p_H)$. For convenience we drop the hat and write the partonic cross section as:

$$\sigma_{ij\to R\,H} = \frac{1}{2s} \int \mathrm{d}\Phi(p_1, p_2; p_3, \ldots, p_H) |M_{ij\to RH}|^2,$$

where $p_1 = x_1 P_1$ and $p_2 = x_2 P_2$ are the momenta of the incoming partons with P_1 and P_2 being the momenta of the protons, $s = 2p_1 \cdot p_2$ denotes the center-of-mass frame energy of the colliding partons and $|M_{ij\to RH}|^2$ is the corresponding squared matrix element. We can rewrite the production phase-space as:

$$\mathrm{d}\Phi(p_1, p_2; p_3, \ldots, p_H) = \mathrm{d}\Phi(p_1, p_2; p_3, \ldots, p_H)\; s\, \mathrm{d}z\, \delta(zs - m_H^2),$$

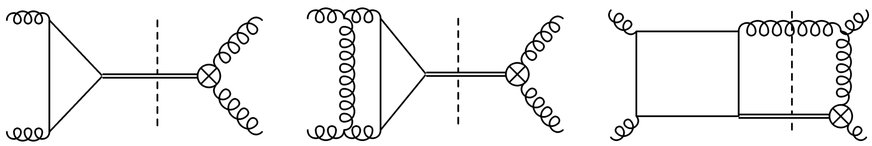


Figure 5.1: Typical contributions of order $y_t y_b$ to the gg-initiated cross-section: Born (left), virtual (middle) and real (right).

where the four vector p_H has the mass-shell condition $p_H^2 = sz$ on the right-hand side and we have $z = m_H^2/s \in [0,1]$. We can then use the mass-shell delta function $\delta(zs - m_H^2)$ to put constraint on s rather than on z and use the latter as an integration variable parametrizing the soft limit of the integral. This amounts to rewriting the total hadronic cross section (5.2) as

$$\sigma^{\text{tot.}}_{P_1P_2\to H} = \sum_{i,j}\sum_{R}\int_\tau^1 \mathrm{d}z\, \mathcal{L}_{ij}(z)\, \sigma_{ij\to RH}(z), \tag{5.3}$$

where $\tau = m_H^2/S$ is the production threshold with $S = 2P_1 \cdot P_2$. $\sigma_{ij\to RH}(z)$ is the partonic cross section with $p_H^2 = zs$. We define the parton luminosity $\mathcal{L}_{ij}(z)$ as

$$\mathcal{L}_{ij}(z) = \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2\, \left(x_1 f_i(x_1)\right)\left(x_2 f_j(x_2)\right)\, \delta\left(z x_1 x_2 - \tau\right). \tag{5.4}$$

This representation allows one to consider the inclusive cross-section as a distribution with respect to the integration variable z and allows the implementation of the soft subtraction without any explicit reference to the luminosity. Note that such a representation requires that we choose z and m_H^2 as independent variables such that we must set $s = m_H^2/z$.

We can decompose the top-bottom interference cross section, $\sigma^{\text{int.}}_{gg\to H}$, into Born, double virtual, and real virtual contributions up to α_s^3 as:

$$\sigma^{\text{int.}}_{ij\to H} = \sigma^{\text{B; int.}}_{ij\to H} + \sigma^{\text{V; int.}}_{ij\to H} + \sum_k \sigma^{\text{R; int.}}_{ij\to kH} + \mathcal{O}(\alpha_s^4), \tag{5.5}$$

where

$$\sigma^{B;\text{int.}}_{ij\to H} = \frac{n_{ij}}{2s}\int \mathrm{d}\Phi_1(p_1,p_2;p_H)\, 2\Re\, \mathcal{A}^{(0)}_{ij\to H}\left(\mathcal{B}^{(0)}_{ij\to H}\right)^*, \tag{5.6}$$

$$\sigma^{\text{R;int.}}_{ij\to kH} = \frac{n_{ij}}{2s}\int \mathrm{d}\Phi_2(p_1,p_2;p_3,p_H)\; 2\Re\,\mathcal{A}^{(0)}_{ij\to kH}\left(\mathcal{B}^{(0)}_{ij\to kH}\right)^*, \tag{5.7}$$

$$\sigma^{\text{V;int.}}_{ij\to H} = \frac{n_{ij}}{2s}\int \mathrm{d}\Phi_1(p_1,p_2;p_H)\; 2\Re\left\{\mathcal{A}^{(1)}_{ij\to H}\left(\mathcal{B}^{(0)}_{ij\to H}\right)^* + \mathcal{A}^{(0)}_{ij\to H}\left(\mathcal{B}^{(1)}_{ij\to H}\right)^*\right\} \tag{5.8}$$

where n_{ij} is the averaging factor and we made the sum over external polarizations and spins implicit. $\mathcal{A}^{(n)}_{ij\to RH}$ indicates the n-th order QCD correction to the matrix element for the production of RH mediated by a bottom quark loop and by $\mathcal{B}^{(n)}_{ij\to RH}$ the n-th order QCD correction to the matrix element for the production of RH in the infinite top-mass effective theory. Note that the Born matrix element $\mathcal{A}^{(0)}_{ij\to RH}$ still contains a bottom quark loop. Typical diagrams contributing to these cross sections are displayed in Figure 5.1.

The analytic expressions for Born and double-virtual contributions are known in the literature [14]. The integrated real contribution has been obtained numerically before []. We presented its first fully analytic computation as an expansion in m_b in ref. [41], which we will also give here. The different channels contributing to $\sigma^{\text{int.}}_{ij\to H}$ are:

- Born and virtual: Only the gg-initiated channels contributes to the Born and virtual cross sections, such that we will only consider $\sigma^{B;\text{int.}}_{gg\to H}$ and $\sigma^{V;\text{int.}}_{gg\to H}$.
- Real: There are new channels contributing to the real cross sections, namely the intial states gg, qg, $\bar{q}g$, gq, $g\bar{q}$, $q\bar{q}$, and $\bar{q}q$. All these contributions can be easily obtained via the use of crossing symmetry from

  $$\sigma^{R;\text{int.}}_{gg\to gH}, \quad \sigma^{R;\text{int.}}_{qg\to qH}, \quad \text{and} \quad \sigma^{R;\text{int.}}_{q\bar{q}\to gH},$$

  and only these will be presented here.

5.2 Results

In this section we present analytic results for the first order in the small-m_b^2 expansion of all the relevant partonic cross sections contributing to $\sigma^{\text{int.}}_{ij\to H}$ up to order α_s^3. We will display here only the leading terms in m_b^2. We will only give expressions for the unrenormalized partonic cross sections, while singling out contributions cancelled by the different renormalization counter terms. A more detailed discussion can be found in reference [14]. The matrix elements and the computation of the relevant regions for the master integrals will be presented in the next two chapters and in the Appendix.

We start with the gg channel. The inclusive Born and double-virtual contributions decouple completely from the $2 \to 1$ phase-space integration as we mentioned in Section 4.1. Therefore integrated one particle phase space contributes an overall pre-factor:

$$\Phi_1(m_H^2) = \int \mathrm{d}\Phi_1(p_1, p_2; p_H) = \frac{2\pi}{s}\delta(1-z). \tag{5.9}$$

Since the analytical results for these two contributions with full dependence in m_b^2 are known [14], we can expand these to check with our results. We use the package `HypExp` [82, 83] to expand Hypergeometric functions in bottom quark mass to this end.

The leading order of the small bottom mass expansion of the Born contribution reads

$$\begin{aligned}
\sigma^{\text{B; int.}}_{gg\to H} = 2\pi\delta(1-z)\,\tilde{\alpha}_s\,\mathcal{B}\Bigg\{ & -4 - 6\zeta_2 + L^2 \\
& + \epsilon\left(-8 + 6\zeta_2 - 6\zeta_3 - 2\zeta_2 L - L^2 - \frac{2L^3}{3}\right) \\
& + \epsilon^2\left(-16 + 14\zeta_2 + 6\zeta_3 + \frac{25\zeta_4}{2} + 2\left(\zeta_2 + \zeta_3\right)L + \frac{3\zeta_2 L^2}{2} + \frac{2L^3}{3} + \frac{L^4}{4}\right) \\
& + \epsilon^3\Bigg(-32 + 28\zeta_2 + \frac{28\zeta_3}{3} - \frac{25\zeta_4}{2} + 15\zeta_2\zeta_3 - 14\zeta_5 - \left(2\zeta_3 + \frac{9\zeta_4}{2}\right)L \\
& \qquad - \frac{1}{6}\left(9\zeta_2 + 8\zeta_3\right)L^2 - \frac{2\zeta_2 L^3}{3} - \frac{L^4}{4} - \frac{L^5}{15}\Bigg) \\
& + \mathcal{O}\left(\epsilon^4\right)\Bigg\} + \mathcal{O}\left(m_b^4\right),
\end{aligned} \tag{5.10}$$

where we defined

$$\mathcal{B} = \frac{c_H}{4v(1-\epsilon)^2 N_A}\frac{m_b^2}{s}, \qquad L = \log\frac{m_b^2}{s} = \log z\frac{m_b^2}{m_H^2} \in \mathbb{R},$$

and

$$\tilde{\alpha}_s = \left(\frac{s}{\mu}\right)^{-\epsilon} S_\epsilon\,\frac{\alpha_s}{4\pi},$$

where $S_\epsilon = \exp\{\epsilon(\log 4\pi - \gamma_E)\}$ and μ is the 't Hooft scale.

Now we can present the results for the real-virtual contribution. Unlike the double-virtual piece, the real-virtual contribution do depend explicitly on z since now the center of mass energy due to initial partons will be different than that of the Higgs boson mass m_H^2, due to the emission of an extra final-state parton . The real matrix element becomes singular in the soft limit $z \to 1$, when the energy of the additional final-state parton vanishes. Therefore, we need to subtract these singularities. We perform this by first rewriting the cross section as:

$$\sigma^{\text{R; int.}}_{gg\to H}(z) = \left(\sigma^{\text{R; int.}}_{gg\to H}(z) - (1-z)^{-1-2\epsilon}\tilde{\sigma}^{\text{R; int.}}_{gg\to H}\right) + (1-z)^{-1-2\epsilon}\tilde{\sigma}^{\text{soft; int.}}_{gg\to H}, \tag{5.11}$$

where $\tilde{\sigma}^{\text{R; int.}}_{gg\to H} = \lim_{z\to 1} \sigma^{\text{R; int.}}_{gg\to H}(z)/(1-z)^{-1-2\epsilon}$ is the soft limit of the real contributions and has the expected form [128]. The term in parenthesis in (5.11) is regular as $z \to 1$ and can be safely expanded in ϵ. The second term on the other hand is expanded by using the plus-prescription method which we described already in Section 4.3:

$$(1-z)^{-1-2\epsilon} \to -\frac{\delta(1-z)}{2\epsilon} + \sum_{n=0}^{\infty}(-2\epsilon)^n \mathcal{D}_n(1-z),$$

where the plus-distributions $\mathcal{D}_n$ are defined as

$$\int \mathrm{d}z\, \mathcal{D}_n(1-z) f(z) = \int \mathrm{d}z\, \log^n(1-z)\, \frac{f(z)-f(1)}{z},$$

for an arbitrary function f.

Now the total next-to-leading order contribution can be written as

$$\begin{aligned}\sigma^{\text{V; int.}}_{gg\to H} + \sigma^{\text{R; int.}}_{gg\to H} = {}& 4\tilde{\alpha}_s \left\{\frac{\beta_0}{2\epsilon} + \frac{3C_F}{4\epsilon}\frac{\partial}{\partial m_b^2}\right\}\sigma^{\text{B; int.}}_{gg\to H} - 8\tilde{\alpha}_s[\,\Delta^{(1)}_{gg} \otimes \sigma^{\text{B; int.}}_{gg\to H}\,] \\ & + 32\pi\tilde{\alpha}_s^2(1-\epsilon)^2 N_c\,\mathcal{B}\left(\delta(1-z)C_\delta + C_+ + C_r + \mathcal{O}(m_b^4)\right),\end{aligned} \tag{5.12}$$

where β_0 is the first order of the QCD beta function when expanded with respect to $\alpha_s/4\pi$, that is

$$\beta_0 = \frac{11C_A - 4T_R N_F}{3},$$

the $g \to gg$ splitting kernel is given by

$$\Delta_{gg}^{(1)}(z) = \frac{\beta_0}{4\epsilon}\,\delta(1-z) + \frac{C_A}{\epsilon}\left(\mathcal{D}_0(1-z) + z(1-z) - 2 + \frac{1}{z}\right),$$

and the convolution is defined for arbitrary functions f and g as

$$[\,f \otimes g\,](z) = \int_0^1 \mathrm{d}x\mathrm{d}y\, f(x)g(y)\delta(xy-z).$$

The first term in (5.12) is cancelled by the α_s and mass renormalizations while the second term is cancelled by the PDF counter terms. We now give the expressions for different pieces in the rest of the expression. δ part is given by

$$\begin{aligned}
C_\delta =& -\frac{7}{2} - \frac{49\zeta_2}{12} + \frac{31\zeta_3}{18} - \frac{307\zeta_4}{48} + \left(\frac{4}{9} - \frac{2\zeta_2}{9} + \frac{11\zeta_3}{18}\right) L \\
&+ \left(\frac{1}{72} + \frac{23\zeta_2}{72}\right) L^2 - \frac{L^3}{9} + \frac{5L^4}{864} \\
&+ \epsilon\Bigg(- \frac{157}{6} - \frac{19\zeta_2}{3} + \frac{307\zeta_3}{36} + \frac{763\zeta_4}{144} + \frac{121\zeta_3\zeta_2}{12} - \frac{5\zeta_5}{18} \\
&\quad + \left(\frac{4}{3} - \frac{\zeta_2}{3} + \frac{31\zeta_3}{18} + \frac{11\zeta_4}{18}\right) L + \left(\frac{85\zeta_2}{72} - \frac{29\zeta_3}{72} - \frac{11}{36}\right) L^2 \\
&+ \left(-\frac{3\zeta_2}{8} - \frac{19}{216}\right) L^3 - \frac{7L^5}{1440} + \frac{13L^4}{108}\Bigg) + \mathcal{O}(\epsilon^2),
\end{aligned}$$

a plus-distribution part given by

$$\begin{aligned}
C_+ =& \left(-4 - 6\zeta_2 + L^2 + \epsilon\left(-16 - 6\zeta_2 - 6\zeta_3 - 2\zeta_2 L + L^2 - 2L^3/3\right)\right)\mathcal{D}_1(1-z) \\
&- \epsilon\left(-4 - 6\zeta_2 + L^2\right)\mathcal{D}_2(1-z) + \mathcal{O}(\epsilon^2),
\end{aligned}$$

and a regular parts given by

$$\begin{aligned}
C_r =& \frac{1}{48z\bar{z}}\Bigg\{4\,\log^3(\bar{z})\,\bar{z}\Big(7\bar{z}^3 - 20\bar{z}^2 + 24\bar{z} - 13\Big) \\
&+ 6\log^2(\bar{z})\,\Big(\bar{z}^2\left(-3\bar{z}^2 + \bar{z} + 2\right) - 2\left(5\bar{z}^4 - 8\bar{z}^3 + 6\bar{z}^2 - 4\bar{z} - 1\right)\log(z)\Big) \\
&+ 4\log(\bar{z})\,\Bigg(+ 3\left(-17\bar{z}^3 + 20\bar{z}^2 - 27\bar{z} + 16\right)\bar{z} + 3\,\log(z)\left(3\bar{z}^2 + \bar{z} - 4\right)\bar{z}^2 \\
&\quad + 3\,\log^2(z)\left(5\bar{z}^3 - 9\bar{z}^2 + 12\bar{z} - 4\right)\bar{z} - 6\,\mathrm{Li}_2(z)\left(2\bar{z}^4 - 2\bar{z}^3 - \bar{z} - 1\right)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \pi^2 \left(-16\bar{z}^4 + 37\bar{z}^3 - 54\bar{z}^2 + 20\bar{z} - 1\right)\Bigg)\\
&- 24\,\mathrm{Li}_3(\bar{z}) \left(\bar{z}^4 + 2\bar{z}^3 - 6\bar{z}^2 + 2\bar{z} - 1\right)\\
&+ 4\,\log^3(z) \left(7\bar{z}^4 - 16\bar{z}^3 + 24\bar{z}^2 - 12\bar{z} + 7\right)\\
&+ 2\,\log^2(z)\,\bar{z}\Big(- 20\bar{z}^3 + 2\bar{z}^2 + 63\bar{z} - 45\Big)\\
&- 4\,\log(z) \Big(21\bar{z}^4 - 76\bar{z}^3 + 127\bar{z}^2 + \pi^2 \left(7\bar{z}^4 - 16\bar{z}^3 + 24\bar{z}^2 - 8\bar{z} + 7\right) - 48\bar{z} + 24\Big)\\
&+ 4\left(25\bar{z}^2 + 26\bar{z} - 51\right)\bar{z}^2 + 3\pi^2\,\bar{z}\Big(9\bar{z}^3 + 3\bar{z}^2 - 22\bar{z} + 10\Big)\\
&+ 12\,\mathrm{Li}_2(z)\,(\bar{z} - 1)\,\bar{z}\left(2\bar{z}^2\log(z) + 15\right)\\
&- 24\,\mathrm{Li}_3(z)\Big(3\bar{z}^3 - 6\bar{z}^2 + 2\bar{z} - 1\Big) + 24\,\zeta_3\left(2\bar{z}^4 - 7\bar{z}^3 + 6\bar{z}^2 - 6\bar{z} - 1\right)\\
&+ L\Bigg(12\,\log^2(\bar{z})\,\bar{z}\Big(- 3\bar{z}^3 + 10\bar{z}^2 - 12\bar{z} + 7\Big)\\
&\qquad\qquad - 24\,\log(\bar{z})\,\bar{z}\Big(\left(2\bar{z}^3 - 3\bar{z}^2 + 6\bar{z} - 1\right)\log(z) + (\bar{z} - 1)\bar{z}\Big)\\
&\qquad\qquad - 24\,\log^2(z)\Big(2\bar{z}^4 - 4\bar{z}^3 + 6\bar{z}^2 - 3\bar{z} + 2\Big) + 4\,\log(z)\,\bar{z}^2\Big(11\bar{z}^2 - 5\bar{z} - 6\Big)\\
&\qquad\qquad + 24\,\mathrm{Li}_2(z)\Big(\bar{z}^4 - \bar{z}^3 - \bar{z} - 1\Big) + 2\,\pi^2\Big(4\bar{z}^4 - 13\bar{z}^3 + 18\bar{z}^2 - 7\bar{z} + 2\Big)\\
&\qquad\qquad + 6\,\left(\bar{z}^2 + 3\bar{z} - 4\right)\bar{z}^2\Bigg)\\
&+ L^2\Bigg(12\,\log(\bar{z})\,\bar{z}\Big(5\bar{z}^3 - 13\bar{z}^2 + 18\bar{z} - 8\Big) + 12\,\log(z)\Big(\bar{z}^4 + 1\Big) - 9(\bar{z} - 1)\bar{z}^3\Bigg)\\
&+ 2\,L^3\,\bar{z}\Big(5\bar{z}^2 - 6\bar{z} + 5\Big)\Bigg\}\\
- \frac{\epsilon}{720 z\bar{z}}&\Bigg\{120\,\log^4(\bar{z})\,\bar{z}\Big(7\bar{z}^3 - 20\bar{z}^2 + 24\bar{z} - 13\Big)\\
&\quad + 30\,\log^3(\bar{z})\Big(4\left(21\bar{z}^4 - 84\bar{z}^3 + 126\bar{z}^2 - 10\bar{z} + 3\right)\log(z)\\
&\qquad\qquad + \bar{z}\left(-41\bar{z}^3 + 37\bar{z}^2 - 10\bar{z} + 18\right)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- 30\ \log^2(\bar{z})\Bigg(- 6\,\mathrm{Li}_2(\bar{z})\,\bar{z}\Big(25\bar{z}^3 - 73\bar{z}^2 + 96\bar{z} - 20\Big)\\
&\qquad - 12\,\mathrm{Li}_2(z)\,\Big(8\bar{z}^4 - 32\bar{z}^3 + 48\bar{z}^2 - 5\bar{z} + 2\Big)\\
&\qquad + 3\ \log^2(z)\Big(8\bar{z}^4 - 33\bar{z}^3 + 42\bar{z}^2 - 4\bar{z} - 5\Big)\\
&\qquad + \pi^2\Big(74\bar{z}^4 - 202\bar{z}^3 + 276\bar{z}^2 - 90\bar{z} + 4\Big)\\
&\qquad + \log(z)\,\Big(- 57\bar{z}^4 + 57\bar{z}^3 - 261\bar{z}^2 + 267\bar{z} + 6\Big)\\
&+ 30\ \log(\bar{z})\,\Bigg(24\,\mathrm{Li}_3(\bar{z})\,\Big(- 3\bar{z}^4 + 6\bar{z}^2 + 3\bar{z} + 1\Big)\\
&\qquad + \log^2(z)\,\bar{z}\,\big(-61\bar{z}^3 + 67\bar{z}^2 - 60\bar{z} + 30\big)\\
&\qquad - 2\ \log(z)\,\Big(\pi^2\,\big(14\bar{z}^3 - 12\bar{z}^2 - 21\bar{z} + 19\big)\\
&\qquad\qquad + 2\,\big(42\bar{z}^4 - 61\bar{z}^3 + 91\bar{z}^2 - 48\bar{z} + 24\big)\,\Big)\\
&\qquad + \pi^2\Big(74\bar{z}^4 - 52\bar{z}^3 - 43\bar{z}^2 + 47\bar{z} + 2\Big)\\
&\qquad + 12\,\zeta_3\Big(44\bar{z}^4 - 119\bar{z}^3 + 150\bar{z}^2 - 52\bar{z} - 7\Big)\\
&\qquad + 2\bar{z}\,\big(230\bar{z}^3 - 23\bar{z}^2 + 81\bar{z} - 192\big)\,\Bigg)\\
&+ 360\,\mathrm{Li}_4(\bar{z})\,\bar{z}\Big(9\bar{z}^3 - 9\bar{z}^2 - 14\Big)\\
&- 360\,S_{2,2}(\bar{z})\,\Big(22\bar{z}^4 - 55\bar{z}^3 + 66\bar{z}^2 - 12\bar{z} - 5\Big)\\
&+ 180\,\mathrm{Li}_2(\bar{z})\log^2(z)\Big(22\bar{z}^4 - 55\bar{z}^3 + 66\bar{z}^2 - 12\bar{z} - 5\Big)\\
&- 180\,\mathrm{Li}_3(\bar{z})\Big(2\,\big(20\bar{z}^4 - 53\bar{z}^3 + 66\bar{z}^2 - 12\bar{z} - 5\big)\log(z)\\
&\qquad + 7\bar{z}^4 + 9\bar{z}^3 - 95\bar{z}^2 + 81\bar{z} + 2\Big)\\
&- 15\ \log^4(z)\,\Big(\bar{z}^4 + 4\bar{z}^3 - 6\bar{z}^2 - 8\bar{z} + 1\Big)\\
&- 10\ \log^3(z)\,\Big(80\bar{z}^4 - 62\bar{z}^3 - 93\bar{z}^2 + 93\bar{z} + 42\Big)\\
&- 10\ \log^2(z)\,\Big(3\pi^2\,\big(39\bar{z}^4 - 97\bar{z}^3 + 129\bar{z}^2 - 27\bar{z} + 9\big)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad - 2\left(77\bar{z}^4 + 163\bar{z}^3 - 645\bar{z}^2 + 297\bar{z} - 36\right)\Big) \\
&+ 360\,\log(z)\,\zeta_3\,\bar{z}\Big(25\bar{z}^3 - 61\bar{z}^2 + 78\bar{z} - 20\Big) \\
&+ 20\,\log(z)\,\Big(\pi^2\left(61\bar{z}^4 - 25\bar{z}^3 - 108\bar{z}^2 + 93\bar{z} + 21\right) \\
&\qquad\qquad + 2\left(195\bar{z}^4 - 646\bar{z}^3 + 901\bar{z}^2 - 324\bar{z} + 162\right)\Big) \\
&+ 360\,\mathrm{Li}_4(z)\,\Big(13\bar{z}^4 - 40\bar{z}^3 + 54\bar{z}^2 - 25\bar{z} + 8\Big) \\
&- 180\,\mathrm{Li}_3(z)\,\Big(2\left(9\bar{z}^4 - 24\bar{z}^3 + 30\bar{z}^2 - 19\bar{z} + 4\right)\log(z) + 9\bar{z}^4 + 5\bar{z}^3 - 76\bar{z}^2 + 64\bar{z} + 2\Big) \\
&+ 60\,\mathrm{Li}_2(z)\,\Big(3\left(6\bar{z}^4 - 15\bar{z}^3 + 18\bar{z}^2 - 11\bar{z} + 2\right)\mathrm{Li}_2(z) \\
&\qquad + 3\left(\bar{z}^3 + 5\bar{z}^2 - 38\bar{z} + 32\right)\bar{z}\log(z) \\
&\qquad + 3\left(29\bar{z}^4 - 71\bar{z}^3 + 84\bar{z}^2 - 27\bar{z} - 3\right)\log^2(z) \\
&\qquad - 6\pi^2\bar{z}^4 + 15\pi^2\bar{z}^3 + 18\bar{z}^3 - 18\pi^2\bar{z}^2 - 222\bar{z}^2 + 21\pi^2\bar{z} + 156\bar{z} - 4\pi^2\Big) \\
&- 360\,S_{2,2}(z)\,\bar{z}\Big(25\bar{z}^3 - 73\bar{z}^2 + 96\bar{z} - 20\Big) \\
+ L\,\Bigg(&- 60\,\log^3(\bar{z})\,\bar{z}\Big(11\bar{z}^3 - 40\bar{z}^2 + 48\bar{z} - 29\Big) \\
&- 180\,\log^2(\bar{z})\,\Big(\left(3\bar{z}^4 - 2\bar{z}^3 + 6\bar{z}^2 + 10\bar{z} - 1\right)\log(z) + \bar{z}\left(-3\bar{z}^3 + 12\bar{z}^2 - 12\bar{z} + 5\right)\Big) \\
&\qquad + 24\,\log^2(z)(\bar{z} - 1) + 36(\bar{z} - 1)\bar{z}^2 + \pi^2\left(7\bar{z}^4 - 22\bar{z}^3 + 30\bar{z}^2 - 8\bar{z} + 1\right)\Big) \\
&+ 360\,\mathrm{Li}_3(\bar{z})\,\Big(\bar{z}^4 - 4\bar{z}^3 + 6\bar{z}^2 - 12\bar{z} + 1\Big) \\
&+ 120\,\log^3(z)\,\Big(3\bar{z}^4 - 6\bar{z}^3 + 9\bar{z}^2 - 7\bar{z} + 3\Big) + 360\,\log^2(z)\,\Big(2\bar{z}^4 - 3\bar{z}^3 + 4\bar{z}^2 - 2\bar{z} + 2\Big) \\
&+ 20\,\log(z)\,\bar{z}\Big(- 91\bar{z}^3 + 46\bar{z}^2 + \pi^2\left(9\bar{z}^3 - 6\bar{z}^2 + 9\bar{z} + 3\right) + 27\bar{z} + 18\Big) \\
&- 540\,\zeta_3\,\bar{z}\Big(2\bar{z}^3 - 5\bar{z}^2 + 6\bar{z} - 7\Big) - 15\pi^2\left(11\bar{z}^4 - 19\bar{z}^3 + 12\bar{z}^2 + 4\right) \\
&+ 360\,\mathrm{Li}_2(z)\,\Big(4\bar{z}\log(z) - \bar{z}^4 + 3\bar{z}^3 - 4\bar{z}^2 + \left(\bar{z}^4 - \bar{z}^3 - 7\bar{z} - 1\right)\log(\bar{z}) + 3\bar{z} + 1\Big) \\
&- 2160\bar{z}\mathrm{Li}_3(z) - 15\left(12\left(\bar{z}^2 + 11\bar{z} - 12\right)\bar{z}^2\right)\Bigg) \\
+ L^2\,\Bigg(&180\,\log^2(\bar{z})\,\bar{z}\Big(3\bar{z}^3 - 8\bar{z}^2 + 12\bar{z} - 5\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- 90 \log(\bar{z}) \Big(4 \left(\bar{z}^4 - 3\bar{z}^3 + 6\bar{z}^2 - \bar{z} - 1\right) \log(z) + \bar{z} \left(13\bar{z}^3 - 25\bar{z}^2 + 30\bar{z} - 14\right) \Big) \\
&+ 360 \,\mathrm{Li}_2(z) \Big(\bar{z}^4 - \bar{z}^3 - 1\Big) - 180 \log(z)^2 \Big(3\bar{z}^4 - 6\bar{z}^3 + 9\bar{z}^2 - 5\bar{z} + 3\Big) \\
&+ 30 \log(z) \Big(5\bar{z}^4 + \bar{z}^3 - 18\bar{z}^2 + 6\bar{z} - 6\Big)\Bigg) \\
+ L^3 &\Bigg(60 \log(\bar{z})\, \bar{z}\Big(11\bar{z}^3 - 26\bar{z}^2 + 36\bar{z} - 15\Big) + 60 \log(z) \Big(\bar{z}^4 + 4\bar{z}^3 - 6\bar{z}^2 + 4\bar{z} + 1\Big) \\
&- 30\, \bar{z} \left(3\bar{z}^3 + \bar{z}^2 - 4\bar{z} + 4\right)\Bigg) \\
+ L^4 &\Bigg(30\bar{z} \left(5\bar{z}^2 - 6\bar{z} + 5\right)\Bigg)\Bigg\}
\end{aligned}
$$

where Li_i and $S_{i,j}$ denote the polylogarithms and the (Nielsen) generalized polylogarithms, respectively.

The double-virtual contribution has the singularity structure as predicted by Catani [128] and can be written as

$$
\sigma^{\mathrm{V;\ int.}}_{gg\to H} = 4\tilde{\alpha}_s \left\{ \frac{\beta_0}{2\epsilon} + \frac{3C_F}{4\epsilon} \frac{\partial}{\partial m_b^2} - \cos(\pi\epsilon) \left(\frac{C_A}{\epsilon^2} + \frac{\beta_0}{2\epsilon} \right) \right\} \sigma^{\mathrm{B;\ int.}}_{gg\to H} + \mathcal{O}(\epsilon^0).
$$

The real-virtual contribution should be finite before integration over the phase-space since the bottom quark mass regulates the internal bottom quark loop. This means that it has only a $1/\epsilon$ pole, which is a result of the additional radiated parton being soft or collinear to the initial state partons. In this limit we know that the cross section should be proportional to the convolution of universal splitting functions with the Born cross section. Therefore we can write up to order ϵ^0:

$$
\sigma^{\mathrm{R;\ int.}}_{gg\to H} = -8\tilde{\alpha}_s [\, \Delta^{(1)}_{gg}|_{\beta_0=0} \otimes \sigma^{\mathrm{B;\ int.}}_{gg\to H}\,] + \mathcal{O}(\epsilon^0).
$$

where $\Delta^{(1)}_{gg}$ is the the collinear splitting kernel.

Let us now continue with the qg and $q\bar{q}$-initiated channels. In this case, we can predict that the real-virtual contributions should be finite in the soft limit, $z \to 1$, since these are already LO processes. Therefore, no soft subtraction is needed. In the case of the qg-initiated channel, the only divergence is of IR nature and cancelled by the PDF counter-terms. The corresponding real-virtual contribution can be written as

$$
\begin{aligned}
\sigma^{\text{R; int.}}_{qg\to qH} =& -4\tilde{\alpha}_s[\,\Delta^{(0)}_{gq} \otimes \sigma^{\text{B; int.}}_{gg\to H}\,] - 2\pi\tilde{\alpha}_s^2 \frac{(1-\epsilon)^2 N_A}{9z}\mathcal{B}\Bigg\{ \\
& + \log^3(\bar{z})\Big(\bar{z}^2+1\Big) - 3\,\log^2(\bar{z})\Big(2\log(z) + (\bar{z}-1)\bar{z}\Big) \\
& + \log(\bar{z})\Big(-6\,\log^2(z)\,\big(\bar{z}^2+1\big) + \pi^2\,\big(6\bar{z}^2+8\big) - 12\,\text{Li}_2(z) + 6\,\big(7\bar{z}^2-7\bar{z}+4\big)\Big) \\
& - 12\,\text{Li}_3(\bar{z}) \\
& + \log^3(z)\Big(-3\bar{z}^2-7\Big) + 9\,\log^2(z)\,z + 6\,\log(z)\Big(\pi^2\,\big(3\bar{z}^2+11\big) + 36\bar{z} + 12\Big) \\
& - 12\,\text{Li}_3(z) + 24\,\text{Li}_2(z)\,z + \zeta_3\,\Big(12\bar{z}^2+24\Big) - \pi^2\,z - 18\bar{z}^2 + 6\bar{z} + 12 \\
& + L\Bigg(-3\,\log^2(\bar{z})\,\Big(\bar{z}^2+1\Big) + 6\,\log(\bar{z})\,\Big(2\,\log(z)\,\big(\bar{z}^2+1\big) + (\bar{z}-1)\bar{z}\Big) \\
& \qquad + 6\,\log^2(z)\Big(\bar{z}^2+2\Big) + 6\,\log(z)\,z + 12\,\text{Li}_2(z) - 2\,\pi^2 - 6\bar{z}^2 + 6\bar{z}\Bigg) \\
& + L^2\Bigg(-3\,\log(\bar{z})\,\Big(\bar{z}^2+1\Big) + \log(z)\,\Big(-3\bar{z}^2-3\Big) - 3\bar{z}^2 + 6\bar{z} - 3\Bigg) \\
& + L^3\Big(-\bar{z}^2-1\Big) \\
& + \epsilon\Bigg\{-2\,\log^4(\bar{z})\,\Big(\bar{z}^2+1\Big) + \log^3(\bar{z})\,\Big(112\,\log(z) + \bar{z}(7\bar{z}-5)\Big) \\
& \qquad + \log^2(\bar{z})\Bigg(84\,\text{Li}_2(\bar{z}) + \log^2(z)\,\Big(-24+6\bar{z}^2+6\Big) + \log(z)\,(60\bar{z}-66) \\
& \qquad\qquad + 126\,\text{Li}_2(z) + \pi^2\,\Big(-5\bar{z}^2-26\Big) + 111\bar{z} - 24 - 75\bar{z}^2\Bigg) \\
& \qquad + \log(\bar{z})\Bigg(84\,\text{Li}_3(\bar{z}) + \log^3(z)\Big(2\bar{z}^2+50\Big) \\
& \qquad\qquad + \log^2(z)\,\Big(-6\bar{z}^2-6\bar{z}\Big) + \log(z)\,\Big(-8\pi^2\bar{z}^2 - 72\bar{z} + 8\pi^2 - 24\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - 96\,\mathrm{Li}_3(z) + \mathrm{Li}_2(z)\ (120\bar{z} - 132) + 96\,\zeta_3 \\
&\qquad + \pi^2 \Big(4\bar{z}^2 - 12\bar{z} + 22\Big) + 234\,\bar{z}^2 - 210\,\bar{z} + 72\Bigg) \\
&- 84\,\mathrm{Li}_4(\bar{z}) + \mathrm{Li}_3(\bar{z})\ \Big(- 132 + 120\bar{z} - 96\log(z)\Big) \\
&+ 48\,\mathrm{Li}_2(\bar{z})\ \log^2(z) - 96\,S_{2,2}(\bar{z}) \\
&+\ \frac{\log^4(z)}{2}\ \Big(5 - \bar{z}^2\Big) + \log^3(z)\ \Big(- 3\bar{z}^2 + 11\bar{z} - 18\Big) \\
&+ \log^2(z)\ \Big(36\,\mathrm{Li}_2(z) + -\frac{1}{4}\,\pi^2\ \Big(13\bar{z}^2 + 93\Big) + 3\ \Big(\bar{z}^2 - 18\bar{z} + 1\Big)\ \Big) \\
&+ \log(z)\ \Bigg(\zeta_3\ \Big(18\bar{z}^2 + 114\Big) + 48\,\mathrm{Li}_2(z)\,z + +\pi^2\ \Big(3\bar{z}^2 - 12\bar{z} + 23\Big) \\
&\qquad + 168\bar{z} + 36 - 12\bar{z}^2\Bigg) \\
&+ 60\,\mathrm{Li}_4(z) - 168\,S_{2,2}(z) + \pi^4\,\frac{1}{60}\ \Big(53\bar{z}^2 - 39\Big) \\
&+ \mathrm{Li}_3(z)\ (96\bar{z} - 108) - 114\,\zeta_3\ \Big(24\bar{z}^2 - 114\bar{z} + 126\Big) \\
&+ \mathrm{Li}_2(z)\ \Big(- 72\bar{z} + 8\pi^2 + 24\Big) + -2\,\pi^2\ \Big(\bar{z}^2 - 11\bar{z} - 1\Big) - 150\bar{z}^2 + 102\bar{z} + 48 \\
&+ L\ \Bigg(\ \log^3(\bar{z})\ \Big(5\bar{z}^2 + 5\Big) + \log^2(\bar{z})\ \Big(- 6\ \Big(2\ \log(z)\ \Big(\bar{z}^2 + 3\Big) + \bar{z}(2\bar{z} - 1)\Big)\ \Big) \\
&\qquad + \log(\bar{z})\ \Big(- 48\,\mathrm{Li}_2(z) + 8\,\pi^2 + \log(z)\ \Big(12\bar{z}^2 + 12\bar{z}\Big) + 24\bar{z}^2 - 24\bar{z}\Big) \\
&\qquad - 48\,\mathrm{Li}_3(\bar{z}) + \log^3(z)\ \Big(3\bar{z}^2 + 1\Big) \\
&\qquad + \log^2(z)\ \Big(6\bar{z}^2 + 12\Big) + \log(z)\ \Big(\frac{5\pi^2\bar{z}^2}{2} - 6\bar{z}^2 + \frac{5\pi^2}{2} + 6\Big) \\
&\qquad - 36\mathrm{Li}_3(z) + \mathrm{Li}_2(z)(24\log(z) + 12) \\
&\qquad + 36\,\zeta_3 + \pi^2\ \Big(2\bar{z}^2 - 3\bar{z} - 1\Big) - 24\bar{z}^2 + 24\bar{z}\Bigg) \\
&+ L^2\ \Bigg(3\ \log^2(\bar{z})\ \Big(\bar{z}^2 + 1\Big) - 3\ \log(\bar{z})\Big(2\ \Big(\bar{z}^2 + 1\Big)\log(z) + \bar{z}(\bar{z} + 1)\Big) \\
&\qquad - \log^2(z)\,\frac{3}{2}\ \Big(3\bar{z}^2 + 5\Big) - 3\ \log(z)\ \Big(\bar{z}^2 - \bar{z} + 2\Big) - 6\,\mathrm{Li}_2(z)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + \pi^2 \, \frac{1}{4} \left(\bar{z}^2 + 5 \right) - 3\, z \Bigg) \\
&\qquad + L^3 \left(3 \, \log(\bar{z}) \left(\bar{z}^2 + 1 \right) + \log(z) \left(2\bar{z}^2 + 2 \right) + 2\bar{z}^2 - 6\bar{z} + 2 \right) \\
&\qquad + L^4 \left(\bar{z}^2 + 1 \right) \Bigg\} \\
&+ \mathcal{O}\left(\epsilon^2 \right) \Bigg\} + \mathcal{O}(m_b^4),
\end{aligned}
$$

where $\bar{z} \equiv 1 - z$ and the $qg \to q$ splitting kernel is given by

$$
\Delta_{gq}^{(1)}(z) = \frac{2}{3\epsilon} \frac{1 + (1-z)^2}{z} \, .
$$

The $q\bar{q}$-initiated channel does not have any IR singularity, since all the contributing diagrams must have the final-state gluon attached to the bottom quark loop, and is therefore completely finite. The corresponding contribution reads

$$
\begin{aligned}
\sigma^{\text{R; int.}}_{q\bar{q}\to gH} = \tilde{\alpha}_s^2 \, \frac{(2(1-\epsilon)^2 N_A)^2}{36} \mathcal{B} \, \Bigg\{ \\
& - \frac{2}{3} \left(\bar{z}^2 \log^2(z) - 4\bar{z} \log(z) - 4\bar{z}^2 \right) + L \, \frac{4}{3} \, \log(z) \, \bar{z}^2 \\
& + \epsilon \, \Bigg\{ \log(\bar{z}) \left(\left(\frac{4}{3} \log^2(z) \, \bar{z}^2 - \frac{16}{3} \, \log(z) \, \bar{z} - \frac{16\bar{z}^2}{3} \right) \right) \\
& \qquad + \frac{2}{9} \bar{z}^2 \log^3(z) + \log^2(z) \left(- \frac{4\bar{z}^2}{9} - \frac{4\bar{z}}{3} \right) \\
& \qquad + \log(z) \left(- \frac{16}{3} \zeta_2 \bar{z}^2 - \frac{8\bar{z}^2}{3} + \frac{88\bar{z}}{9} \right) + \frac{88\bar{z}^2}{9} \\
& \qquad + L \left(- \frac{8}{3} \, \log(\bar{z}) \, \bar{z}^2 \log(z) + \frac{8}{9} \, \log(z) \, \bar{z}^2 \right) \\
& \qquad + L^2 \left(- \frac{2}{3} \, \log(z) \, \bar{z}^2 \right) \Bigg\} \\
& + \mathcal{O}(\epsilon^2) \Bigg\} + \mathcal{O}(m_b^4).
\end{aligned} \tag{5.13}
$$

We presented here our results up to order ϵ since these terms will be useful for the renormalization of the finite bottom quark mass effects at NNLO.

Chapter 6

Real Virtual Contribution

In this chapter we will present the results for the real-virtual contribution. In the first part we will explain how we calculated matrix elements in terms of master integrals, and in the second part we will describe how we calculated different scalings of master integrals and subsequently boundary conditions themselves.

We compute the inclusive real contributions using the method of *reverse unitarity* as explained in Section 4.2. We first generate the Feynman diagrams via FeynArts [129]. And use our own Mathematica code to dress the diagrams, i.e. assign momenta, spin and colour indices, as well as applying the Feynman rules and contracting the indices. The spinor/colour traces are evaluated via FeynCalc [130]. The calculation is performed in Feynman gauge and we therefore add contributions with external Faddeev-Popov ghosts.

6.1 Matrix Element Squared

The QCD corrections to the Higgs production via real emission involves partons in the final state, which means we have to take into account the processes involving the light quarks in the initial and final states.

Gluon Channel

We first consider the process $g(p_1) + g(p_2) \to H(p_H) + g(p_3)$ in the full theory with bottom quarks running in the loop and interfere it with the same process in the effective theory. We recall from Chapter 5 that the real-virtual contribution to the cross section is given by:

$$\sigma^{\mathrm{R;int.}}_{ij\to kH} = \frac{n_{ij}}{2s} \int \mathrm{d}\Phi_2(p_1,p_2;p_3,p_H)\; 2\Re\,\mathcal{A}^{(0)}_{ij\to kH} \left(\mathcal{B}^{(0)}_{ij\to kH}\right)^* \tag{6.1}$$

where $\mathcal{A}^{(0)}_{gg\to gH}$ indicate the matrix element contribution at leading order to the process $gg \to gH$ in the full theory with bottom quarks running in the loop, whereas $\mathcal{B}^{(0)}_{gg\to gH}$ is the contribution in the effective theory. Φ_2 is the two particle phase space. As stated before we calculate the squared matrix element for real emission processes with reverse unitarity. Then we can write the squared matrix element integrated over phase space as:

$$\int \mathrm{d}\Phi_2(p_1,p_2;p_3,p_H) \sum_{\mathrm{pols,cols}} 2\Re\,\mathcal{A}^{(0)}_{gg\to gH} \left(\mathcal{B}^{(0)}_{gg\to gH}\right)^* = -\frac{c_H\, g_s^4 N_c \left(N_c^2-1\right)}{v} A_{\mathrm{R}}, \tag{6.2}$$

where c_H is the Wilson coefficient, g_s is the strong coupling, and v is the vacuum expectation value of the Higgs field; and the summation over the polarizations and colors of the gluons is explicitly written. Here, we used the formulation we introduced in Chapter 4.5 to obtain the interference of the two amplitudes at the right hand side of Equation 6.2. We find two topologies for this channel and after the reduction 16 master integrals which are depicted as figures in the next section. We present A_{R} in terms of master integrals in the Appendix, since it is rather lengthy. We use the parameter $r = m_b^2/s$ to express our results for the matrix elements.

Quark Channels

For the real emission involving light quarks we consider the following channels:

$$q(p_1) + \bar{q}(p_2) \to H(p_H) + g(p_3) \tag{6.3}$$
$$q(p_1) + g(p_2) \to H(p_H) + q(p_3) \tag{6.4}$$
$$q(p_2) + g(p_1) \to H(p_H) + q(p_3) \tag{6.5}$$
$$\bar{q}(p_1) + g(p_2) \to H(p_H) + \bar{q}(p_3) \tag{6.6}$$
$$\bar{q}(p_2) + g(p_1) \to H(p_H) + \bar{q}(p_3) \tag{6.7}$$

In the first channel quarks are in the initial state, whereas the other channels have quarks both in initial and final states. The latter channels have the same cross section

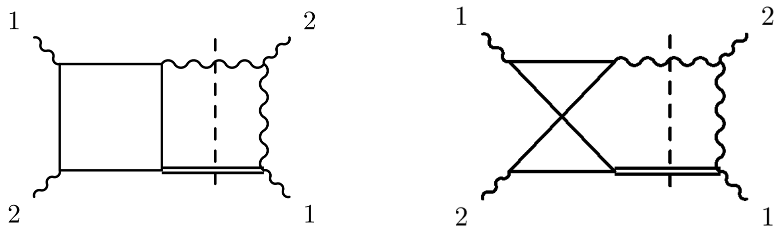


Figure 6.1: The two independent topologies appearing in the computation of the real contributions.

since the cross section is invariant under the exchange of two initial state momenta, p_1 and p_2. Therefore we present only one result below.

Similar to the gluon channel we write the squared matrix element as the following:

$$\int \mathrm{d}\Phi_2(p_1, p_2; p_3, p_H) \sum_{\text{pols,cols}} 2\Re\, \mathcal{A}^{(0)}_{q\bar{q}\to gH} \left(\mathcal{B}^{(0)}_{q\bar{q}\to gH}\right)^* = -\frac{c_H\, g_s^4 N_c \left(N_c^2 - 1\right)}{v} A_{q\bar{q}} \tag{6.8}$$

We use the parameter $r = m_b^2/s$ to express our results for the matrix elements. We present $A_{q\bar{q}}$ in the Appendix.

6.2 Master Integrals

In this section we will describe how we calculated the necessary regions of the master integrals displayed in the figure below. First let us set the notation. We find two distinct topologies here, which we will denote as $T_1^{\mathrm{R}}[\nu_1, \ldots, \nu_5]$ and $T_2^{\mathrm{R}}[\nu_1, \ldots, \nu_5]$, which are depicted in Figure 6.1. We will define these as:

$$T_1^{\mathrm{R}}[\nu_1, \ldots, \nu_5] = \mathrm{e}^{\epsilon\gamma_E} \int \frac{d^d k}{i\pi^{d/2}} \frac{\mathrm{d}\Phi_{12\to Hg}}{D_{11}^{\nu_1} D_{12}^{\nu_2} D_{13}^{\nu_3} D_{14}^{\nu_4} D_{15}^{\nu_5}} \tag{6.9}$$

where

$$\begin{aligned} D_{11} &= k^2 - m_b^2 & D_{14} &= (k - p_3)^2 - m_b^2 \\ D_{12} &= (k + p_1)^2 - m_b^2 & D_{15} &= (p_2 + p_3)^2 \\ D_{13} &= (k + p_1 + p_2)^2 - m_b^2 \end{aligned} \tag{6.10}$$

$$T_2^{\mathrm{R}}[\nu_1,\ldots,\nu_5] = \mathrm{e}^{\epsilon\gamma_E}\int \frac{d^dk}{i\pi^{d/2}}\frac{\mathrm{d}\Phi_{12\to Hg}}{D_{21}^{\nu_1}D_{22}^{\nu_2}D_{23}^{\nu_3}D_{24}^{\nu_4}D_{25}^{\nu_5}} \tag{6.11}$$

where

$$\begin{aligned} D_{21} &= k^2 - m_b^2 & D_{24} &= (k-p_3)^2 - m_b^2 \\ D_{22} &= (k+p_1)^2 - m_b^2 & D_{25} &= (p_2+p_3)^2 \\ D_{23} &= (k-p_2-p_3)^2 - m_b^2 \end{aligned} \tag{6.12}$$

where k is the loop momentum, p_1 and p_2 are the momenta of the incoming gluons and p_3 is the cut momentum corresponding to the real emission. $\mathrm{d}\Phi_{12\to Hg} = \mathrm{d}\Phi_2(p_1,p_2;p_3,p_H)$ denotes the $2\to 2$ phase space: We parametrize it as

$$\int \mathrm{d}\Phi_2(p_1,p_2;p_3,p_H) = \mathcal{N}\int_0^1 \mathrm{d}\lambda\,(\lambda(1-\lambda))^{-\epsilon}, \quad \text{with} \quad \mathcal{N} = \frac{s^{-\epsilon}\Omega_{d-2}}{4(2\pi)^{d-2}}(1-z)^{1-2\epsilon}, \tag{6.13}$$

with invariants given by

$$s_{13} = (p_1-p_3)^2 = -s\,(1-z)\,\lambda, \qquad s_{23} = (p_2-p_3)^2 = -s\,(1-z)\,(1-\lambda). \tag{6.14}$$

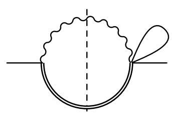
After the reduction we find 16 master integrals:

We obtain the differential equations with respect to the parameter $r = m_b^2/s$, as an expansion around $r=0$. The reduction is performed as usual. Then we find the solutions to the differential equations using the method described in Section 3.3.3. After obtaining the solutions, we proceed as follows: we start from the simplest solution where the right hand side involves only one boundary condition with a specific region. This way we can directly determine the boundary condition by matching it to the corresponding region for the master integral. For example the solution for the first master integral, M_1^{R}, is very trivial and reads:

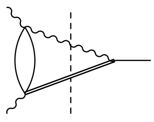
$$M_1^{\mathrm{R}} = r^{1-\epsilon}\,BC_1^{\mathrm{R}} \tag{6.15}$$

Therefore we can first start computing the left hand-side by for example Feynman parametrization, and then relate it to the right hand side to obtain an expression for the boundary condition $\mathrm{BC}_1^{\mathrm{R}}$. Then we can move on to the next simplest solution, which is the one for M_2^{R}:

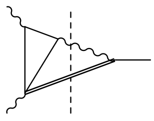
$$M_2^{\mathrm{R}} = r^{1-\epsilon}\,(\ldots)\,BC_1^{\mathrm{R}} + r^{-2\epsilon}\left((\ldots)BC_1^{\mathrm{R}} + (\ldots)BC_2^{\mathrm{R}} + (\ldots)BC_3^{\mathrm{R}}\right)$$



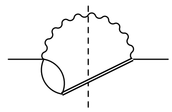
$$= T_1^{\mathrm{R}}[0,0,1,0,0] = M_1^{\mathrm{R}},$$



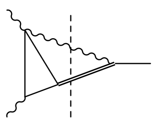
$$= T_1^{\mathrm{R}}[0,1,0,1,0] = M_2^{\mathrm{R}},$$



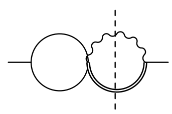
$$= T_1^{\mathrm{R}}[0,1,1,1,0] = M_3^{\mathrm{R}},$$



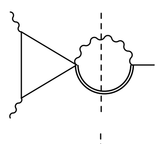
$$= T_1^{\mathrm{R}}[1,0,0,1,0] = M_4^{\mathrm{R}},$$



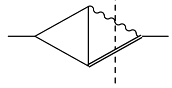
$$= T_1^{\mathrm{R}}[1,1,0,1,0] = M_5^{\mathrm{R}},$$



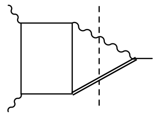
$$= T_1^{\mathrm{R}}[1,0,1,0,0] = M_6^{\mathrm{R}},$$



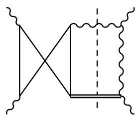
$$= T_1^{\mathrm{R}}[1,1,1,0,0] = M_7^{\mathrm{R}},$$



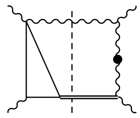
$$= T_1^{\mathrm{R}}[1,0,1,1,0] = M_8^{\mathrm{R}},$$



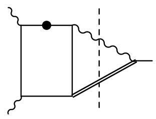
$$= T_1^{\mathrm{R}}[1,1,1,1,0] = M_9^{\mathrm{R}},$$



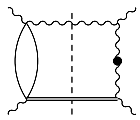
$$= T_1^{\mathrm{R}}[1,1,1,1,1] = M_{10}^{\mathrm{R}},$$



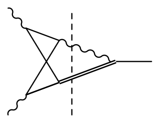
$$= T_1^{\mathrm{R}}[1,1,0,1,-1] = M_{11}^{\mathrm{R}},$$



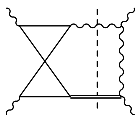
$$= T_1^{\mathrm{R}}[1,1,-1,1,0] = M_{12}^{\mathrm{R}},$$



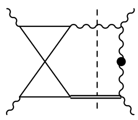
$$= T_1^{\mathrm{R}}[0,1,0,1,-1] = M_{13}^{\mathrm{R}},$$



$$= T_2^{\mathrm{R}}[1,1,1,1,0] = M_{14}^{\mathrm{R}},$$



$$= T_2^{\mathrm{R}}[1,1,1,1,1] = M_{15}^{\mathrm{R}},$$



$$= T_2^{\mathrm{R}}[1,1,1,1,-1] = M_{16}^{\mathrm{R}}.$$

Figure 6.2: Master Integrals that Appear in Real-Virtual Contribution

$$
\begin{aligned}
&+ \frac{-2+3\epsilon}{\epsilon\left(\epsilon^2-1\right)(1-z)^4}\Big(3\epsilon^3 r\,\left(8r^2-r(1-z)+2\,(1-z)^2\right)\\
&\quad+\epsilon\left(9r^3-3r^2\,(1-z)-4r\,(1-z)^2-(1-z)^3\right)\\
&\quad+\epsilon^2\left(-39r^3+12r^2(1-z)+r(1-z)^2-(1-z)^3\right)\\
&\quad+r(1-z)^2+18\epsilon^4 r^2(2r-1+z)\Big)BC_3^{\mathrm{R}}
\end{aligned}
\tag{6.16}
$$

where by using ellipses we refrained from displaying the full solution just to give the basic idea. From the solution above, we see that if we compute the region r^0 of the master integral M_2^{R} first, than we can match it to the right hand side to determine BC_3^{R}. With the analytic expressions for BC_1^{R} and BC_3^{R} at hand, we can also determine BC_2^{R} from calculating the $r^{-2\epsilon}$ region for the same master integral.

This way, we can figure out which regions of the master integrals we need to compute to determine all the boundary conditions. Most of the time, they can be determined by more than one way, which we use as a check in the end. Here we will describe how to compute the regions. All boundary conditions are presented in Appendix C.2. Following ref. [41], the solutions to the differential equations, i.e. the master integrals written in terms of boundary conditions, can be found in Appendix B.2, together with the analytic expressions for masters which are given in Appendix D.2 for completeness.

Now let us describe how we computed these necessary regions from the Mellin Barnes representation. For convenience we drop the factor:

$$
e^{\epsilon\gamma_E}
$$

which we will pick up when we expand the expressions in ϵ.

6.2.1 $M_1^{\mathbf{R}}$

As seen from the figure this is the simplest master integral that we have. The Feynman parameters representation for this integral is:

$$
M_1^{\mathrm{R}} = -\int d\Phi_2 \int_0^1 dx_1 \delta\left(1-x_1\right) m_b^{2-2\epsilon}\frac{\Gamma\left(-1+\epsilon\right)}{x_1}
\tag{6.17}
$$

The integration over the Feynman parameter is trivial due to delta function, and then we directly integrate over the phase space, obtaining:

$$M_1^{\mathrm{R}} = -m_b^{2-2\epsilon}\,\frac{\Gamma(1-\epsilon)^2\,\Gamma(1+\epsilon)}{\epsilon\,(-1+\epsilon)\Gamma(2-2\epsilon)} \tag{6.18}$$

6.2.2 $M_2^{\mathbf{R}}$, $M_4^{\mathbf{R}}$, $M_6^{\mathbf{R}}$, $M_{13}^{\mathbf{R}}$

We start again with the Feynman parametrised integral:

$$\begin{aligned}
M_2^{\mathrm{R}} =& \Gamma(\epsilon)\int d\Phi_2 \int_0^1 dx_1\,dx_2\,\delta(1-x_1-x_2)\,(x_1+x_2)^{-2+2\epsilon}\\
&\times\left(-s_{13}x_1x_2+m_b^2\,(x_1+x_2)^2\right)^{-\epsilon} &(6.19)\\
=& \Gamma(\epsilon)\int d\Phi_2\int_0^\infty dx_2\;(1+x_2)^{-2+2\epsilon}\left(-s_{13}x_2+m_b^2\,(1+x_2)^2\right)^{-\epsilon} &(6.20)\\
=& \int d\Phi_2\int_0^\infty dx_2\int_{-i\infty}^{+i\infty} dz_0\;(-s_{13})^{-\epsilon-z_0}\;m_b^{2z_0}\,\Gamma(\epsilon+z_0)\,\Gamma(-z_0)\\
&\times (x_2)^{-\epsilon-z_0}\,(1+x_2)^{-2+2\epsilon+2z_0} &(6.21)\\
=& \int d\Phi_2\int_{-i\infty}^{+i\infty} dz_0\;(-s_{13})^{-\epsilon-z_0}\;m_b^{2z_0}\frac{\Gamma(1-\epsilon-z_0)^2\,\Gamma(-z_0)\,\Gamma(\epsilon+z_0)}{\Gamma(-2(-1+\epsilon+z_0))} &(6.22)\\
=& \int d\Phi_2\int_{-i\infty}^{+i\infty} dz_0\;(s_{12})^{-\epsilon-z_0}\,(1-z)^{-\epsilon-z_0}\,\lambda^{-\epsilon-z_0}\,m_b^{2z_0}\\
&\times\frac{\Gamma(1-\epsilon-z_0)^2\,\Gamma(-z_0)\,\Gamma(\epsilon+z_0)}{\Gamma(-2(-1+\epsilon+z_0))} &(6.23)\\
=& \int_0^1 d\lambda\int_{-i\infty}^{+i\infty} dz_0\;(1-z)^{-\epsilon-z_0}\,(1-\lambda)^{-\epsilon}\,\lambda^{-2\epsilon-z_0}\,m_b^{2z_0}\\
&\times\frac{\Gamma(1-\epsilon-z_0)^2\,\Gamma(-z_0)\,\Gamma(\epsilon+z_0)}{\Gamma(-2(-1+\epsilon+z_0))} &(6.24)\\
=& \int_{-i\infty}^{+i\infty} dz_0\,m_b^{2z_0}\,(1-z)^{-\epsilon-z_0}\,\Gamma(1-\epsilon)\\
&\times\frac{\Gamma(1-2\epsilon-z_0)\,\Gamma(1-\epsilon-z_0)^2\,\Gamma(-z_0)\,\Gamma(\epsilon+z_0)}{\Gamma(2-3\epsilon-z_0)\,\Gamma(-2(-1+\epsilon+z_0))} &(6.25)
\end{aligned}$$

From first to second line we set the variable x_1 to zero and changed the integration boundary of x_2 to zero to infinity using the Cheng-Wu theorem. Then in order to factor the mass term we introduced a Mellin Barnes integration variable z_0 using Equation 3.75. Doing this also puts the equation into the form of 3.74 so that we can easily integrate over x_2. in the fifth line we introduced the parametrisation for the invariant s_{13} using 4.9. Then we also explicitly wrote the parametrisation of the phase space integration and integrated over λ. Now we have only the Mellin Barnes integration left to be performed.

Note that the exponent of the scale m_b^2 depends on the Mellin Barnes variable z_0. This crucial fact will allow us to obtain different scalings through taking the residues, which will get clearer below.

Let us now look at the Mellin Barnes integral more closely. The procedure we will use here for the Mellin Barnes integrations will be used for the other examples that we will give in this section as well, and therefore will not be repeated. We choose the integration contour in a standard way such that the poles of $\Gamma(\epsilon + z_0)$ are separated from that of $\Gamma(-z_0 + \cdots)$; namely at $-1 < \Re(z_0) < 0$. The contour is closed to the right at infinity and therefore we will take the residues to the right of the integration contour. The reason why we would like to take the residues to the right will become clear later. We should also note that since $\epsilon \to 0$, the pole from $\Gamma(\epsilon + z_0)$ corresponding to $\Re(z_0) = -\epsilon$ might remain actually on the right depending on whether $\Re(\epsilon) < 0$ or $\Re(\epsilon) > 0$. Therefore we might have to subtract the residue at this point. The residue due to this point is:

$$-\frac{m_b^{-2\epsilon}\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(2-2\epsilon)} \tag{6.26}$$

We check that indeed this is the case by comparing our result to a second calculation that is done numerically. Now we can take the remaining residues:

- Due to $\Gamma(-z_0)$:

$$\begin{aligned}&\frac{(1-z)^{-\epsilon}\Gamma(1-2\epsilon)\Gamma(1-\epsilon)^3\Gamma(\epsilon)}{\Gamma(2-3\epsilon)\Gamma(2-2\epsilon)}\\&-\frac{m_b^2(1-z)^{-1-\epsilon}\Gamma(1-\epsilon)\Gamma(-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-3\epsilon)}\end{aligned} \tag{6.27}$$

- Due to $\Gamma(1-2\epsilon-z_0)$: $m_b^{-4\epsilon}$ scaling:

$$\frac{m_b^{2-4\epsilon}\,(1-z)^{-1+\epsilon}\Gamma(1-\epsilon)\Gamma(\epsilon)^2\Gamma(-1+2\epsilon)}{\Gamma(2\epsilon)} \tag{6.28}$$

- Due to $\Gamma(1-\epsilon-z_0)$: $m_b^{-2\epsilon}$ scaling:

$$- \frac{m_b^{-2\epsilon}\Gamma\left(1-\epsilon\right)^2\Gamma\left(\epsilon\right)}{\Gamma\left(2-2\epsilon\right)} + \frac{2m_b^{2-2\epsilon}\,\Gamma\left(1-\epsilon\right)\Gamma\left(-1+\epsilon\right)\Gamma\left(-\epsilon\right)}{\left(1-z\right)\Gamma\left(1-2\epsilon\right)} \tag{6.29}$$

Note that we need only finite amount of residues since we would like to have an expression up to order m^2. Another point worth mentioning is that if we were to close the contour to the left and take the residues to the left, we would end up with expressions with scalings such as $m_b^{-2-2\epsilon}$. However these regions are valid only for masses much larger than the center of mass energy, i.e. $m/s \gg 1$. And since in our case we are interested in the bottom quark contribution we would like our expansion to be around the small mass: $m/s \ll 1$.

Finally, matching these scalings with the solutions for our differential equations, which is given in Equation 6.16, we determine the boundary conditions BC_2^{R} and BC_3^{R}, the analytical expressions of which are presented in the Appendix.

The computation of the masters M_4^{R} and M_6^{R} are similar to M_2^{R}:

$$\begin{aligned} M_4^{\mathrm{R}} =& \Gamma\left(\epsilon\right)\int d\Phi_2 \int_0^1 dx_1\, dx_2\, \delta\left(1-x_1-x_2\right) \\ & \times \left(x_1+x_2\right)^{-2+2\epsilon}\left(-\left(s_{12}+s_{13}+s_{23}\right)x_1x_2 + m_b^2\left(x_1+x_2\right)^2\right)^{-\epsilon} \end{aligned} \tag{6.30}$$

$$\begin{aligned} =& \int d\Phi_2 \int_0^\infty dx_2 \int_{-i\infty}^{+i\infty} dz\, m_b^{2z_0}\, \left(-s_{12}-s_{13}-s_{23}\right)^{-\epsilon-z_0} \\ & \times \Gamma\left(-z_0\right)\Gamma\left(\epsilon+z_0\right)x_2^{-\epsilon-z_0}\left(1+x_2\right)^{-2+2\epsilon+2z_0} \end{aligned} \tag{6.31}$$

$$\begin{aligned} =& \int_{-i\infty}^{+i\infty} dz\, \left(e^{i\pi}\right)^{-\epsilon-z_0}\, m_b^{2z_0}\, z^{-\epsilon-z_0} \\ & \times \frac{\Gamma\left(1-\epsilon\right)^2\Gamma\left(1-\epsilon-z_0\right)^2\Gamma\left(-z_0\right)\Gamma\left(\epsilon+z_0\right)}{\Gamma\left(2-2\epsilon\right)\Gamma\left(2-2\epsilon-2z_0\right)} \end{aligned} \tag{6.32}$$

Here we set $x_1 = 1$ at the beginning and used the Cheng-Wu theorem as usual. The analytic continuation of $(-s)$ gives a factor of $e^{i\pi}$. To compute the Mellin Barnes integral we follow the same procedure as before. For this master integral we do not need the residues due to all the gamma functions that appear, but $\Gamma\left(-z_0\right)$. The reason is that this integral will help us determine the boundary condition BC_4^{R}, and for this we will not need all the scalings, but we only need the m_b^2 piece, of whose contribution comes from the residues due to $\Gamma\left(-z_0\right)$. The residue at zeroth order in

m_b^2 is:

$$\frac{e^{-i\pi\epsilon}z^{-\epsilon}\Gamma\left(1-\epsilon\right)^4\Gamma\left(\epsilon\right)}{\Gamma\left(2-2\epsilon\right)^2} \tag{6.33}$$

The computation of the regions of M_{13}^{R} is very similar to that of M_2^{R}, since it has an extra numerator that can be easily integrated over the phase space, therefore we will skip presenting it here.

6.2.3 $M_3^{\mathbf{R}}$, $M_7^{\mathbf{R}}$

$$\begin{aligned}
M_3^{\mathrm{R}} =& -\Gamma\left(\epsilon+1\right)\int d\Phi_2\int_0^1 dx_1\,dx_2\,dx_3\,\delta\left(1-x_1-x_2-x_3\right)\\
&\times\left(x_1+x_2+x_3\right)^{-1+2\epsilon}\left(-s_{13}x_1x_2+m_b^2\left(x_1+x_2+x_3\right)^2\right)^{-1-\epsilon} &(6.34)\\
=& -\int d\Phi_2\int_0^\infty dx_2\,dx_3\int_{-i\infty}^{+i\infty} dz_0\,m_b^{2z_0}\left(-s_{13}\right)^{-1-\epsilon-z_0}x_2^{-1-\epsilon-z_0}\\
&\times\left(1+x_2+x_3\right)^{-1+2\epsilon+2z_0}\Gamma\left(-z_0\right)\Gamma\left(1+\epsilon+z_0\right) &(6.35)\\
=& -\int d\Phi_2\int_0^\infty dx_3\int_{-i\infty}^{+i\infty} dz_0\,m_b^{2z_0}\left(-s_{13}\right)^{-1-\epsilon-z_0}\left(1+x_3\right)^{-1+\epsilon+z_0}\\
&\times\Gamma\left(-\epsilon-z_0\right)\Gamma\left(1-\epsilon-z_0\right)\Gamma\left(-z_0\right)\Gamma\left(1+\epsilon+z_0\right) &(6.36)\\
=& -\int_0^1 d\lambda\int_{-i\infty}^{+i\infty} dz_0\,m_b^{2z_0}\left(1-z\right)^{-1-\epsilon-z_0}\left(1-\lambda\right)^{-\epsilon}\lambda^{-1-2\epsilon-z_0}\\
&\times\frac{\Gamma\left(-\epsilon-z_0\right)^2\Gamma\left(-z_0\right)\Gamma\left(1+\epsilon+z_0\right)}{\Gamma\left(1-2\epsilon-2z_0\right)} &(6.37)\\
=&-,\Gamma\left(1-\epsilon\right)\int_{-i\infty}^{+i\infty} dz_0\,m_b^{2z_0}\left(1-z\right)^{-1-\epsilon-z_0}\\
&\times\frac{\Gamma\left(-2\epsilon-z_0\right)\Gamma\left(-\epsilon-z_0\right)^2\Gamma\left(-z_0\right)\Gamma\left(1+\epsilon+z_0\right)}{\Gamma\left(1-2\epsilon-2z_0\right)\Gamma\left(1-3\epsilon-z_0\right)} &(6.38)
\end{aligned}$$

Here we followed similar steps to the case for M_2^{R}, so we refrain from explaining the procedure step by step. Now we can discuss the Mellin Barnes integration. The residues are:

- Due to $\Gamma\left(-z_0\right)$:

$$-\frac{\left(1-z\right)^{-1-\epsilon}\Gamma\left(1-\epsilon\right)\Gamma\left(-2\epsilon\right)\Gamma\left(-\epsilon\right)^2\Gamma\left(1+\epsilon\right)}{\Gamma\left(1-3\epsilon\right)\Gamma\left(1-2\epsilon\right)} \tag{6.39}$$

- Due to $\Gamma(-\epsilon - z_0)$: $m_b^{-2\epsilon}$ scaling:

$$-\frac{m_b^{-2\epsilon}\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(\epsilon)}{\Gamma(1-2\epsilon)(1-z)}\left(\gamma_E - \log(m_b^2) + \log(1-z) - \psi^0(1-2\epsilon) + \psi^0(-\epsilon) + \psi^0(\epsilon)\right) \tag{6.40}$$

- Due to $\Gamma(-2\epsilon - z_0)$: $m_b^{-4\epsilon}$ scaling:

$$-\frac{m_b^{-4\epsilon}(1-z)^{-1+\epsilon}\Gamma(1-\epsilon)\Gamma(\epsilon)^2\Gamma(2\epsilon)}{\Gamma(1+2\epsilon)} \tag{6.41}$$

The calculation of M_7^{R} is very similar to the M_3^{R}. The result before the Mellin Barnes integration is:

$$\begin{aligned} M_7^{\mathrm{R}} =& -\Gamma(\epsilon+1)\int d\Phi_2 \int_0^1 dx_1\, dx_2\, dx_3\, \delta(1-x_1-x_2-x_3) \\ & \times (x_1+x_2+x_3)^{-1+2\epsilon}\left(-s_{12}x_1x_3 + m_b^2(x_1+x_2+x_3)^2\right)^{-1-\epsilon} \qquad (6.42)\\ =& -\int_{-i\infty}^{+i\infty} dz_0 \frac{\left(e^{i\pi}\right)^{-1-\epsilon-z_0} m_b^{2z_0}\,\Gamma(1-\epsilon)^2\,\Gamma(-\epsilon-z_0)^2\,\Gamma(-z_0)\,\Gamma(1+\epsilon+z_0)}{\Gamma(2-2\epsilon)\,\Gamma(1-2\epsilon-2z_0)} \qquad (6.43) \end{aligned}$$

The $m_b^{-2\epsilon}$ region of this master integral allows us to determine the boundary condition BC_7^{R}; so we will only focus on the residues contributing to this scaling. which should be generated by the residues due to $\Gamma(-\epsilon - z_0)$. At first order in $m_b^{-2\epsilon}$ we get:

$$-\frac{m_b^{-2\epsilon}e^{-i\pi}\Gamma(1-\epsilon)^2\Gamma(\epsilon)\left(\gamma_E + i\pi - \log(m_b^2) + \psi^{(0)}(\epsilon)\right)}{\Gamma(2-2\epsilon)} \tag{6.44}$$

6.2.4 $M_5^{\mathbf{R}}$, $M_8^{\mathbf{R}}$, $M_{11}^{\mathbf{R}}$

$$\begin{aligned} M_5^{\mathrm{R}} =& -\Gamma(\epsilon+1)\int d\Phi_2 \int_0^1 dx_1\, dx_2\, dx_3\, \delta(1-x_1-x_2-x_3)\,(x_1+x_2+x_3)^{-1+2\epsilon} \\ & \times \left(m_b^2(x_1+x_2+x_3)^2 - x_2\left(s_{12}x_1 + s_{23}x_1 + s_{13}(x_1+x_3)\right)\right)^{-1-\epsilon} \qquad (6.45)\\ =& -\int d\Phi_2 \int_0^\infty dx_3 \int_{-i\infty}^{+i\infty} dz_0\, m_b^{2z_0}\,(1+x_3)^{-1+\epsilon+z_0} \\ & \times \Big(-s_{12} - s_{23} - s_{13}(1+x_3)\Big)^{-1-\epsilon-z_0} \\ & \times \frac{\Gamma(-\epsilon-z_0)\,\Gamma(1-\epsilon-z_0)\,\Gamma(-z_0)\,\Gamma(1+\epsilon+z_0)}{\Gamma(1-2\epsilon-2z_0)} \qquad (6.46) \end{aligned}$$

$$
\begin{aligned}
&= - \int d\Phi_2 \int_{-i\infty}^{+i\infty} dz_0 \int_{-i\infty}^{+i\infty} dz_1\, m_b^{2z_0}\ (s_{13})^{-\epsilon - z_0 + z_1} \Big(- s_{12} - s_{13} - s_{23} \Big)^{-1-z_1} \\
&\quad \times \frac{\Gamma(-z_0)\,\Gamma(-\epsilon - z_0)\,\Gamma(\epsilon + z_0 - z_1)\,\Gamma(-z_1)\,\Gamma(1+z_1)\,\Gamma(1-\epsilon-z_0+z_1)}{\Gamma(1-2\epsilon-2z_0)}
\end{aligned} \tag{6.47}
$$

$$
\begin{aligned}
&= - \ \Gamma(1-\epsilon) \int_{-i\infty}^{+i\infty} dz_0 \int_{-i\infty}^{+i\infty} dz_1\, \left(e^{i\pi}\right)^{-1-z_1}\ m_b^{2z_0}\, z^{-1-z_1}\,(1-z)^{-\epsilon-z_0+z_1} \\
&\quad \times \frac{\Gamma(-z_0)\,\Gamma(-\epsilon-z_0)}{\Gamma(2-3\epsilon-z_0+z_1)} \\
&\quad \times \frac{\Gamma(\epsilon+z_0-z_1)\,\Gamma(-z_1)\,\Gamma(1+z_1)\,\Gamma(1-2\epsilon-z_0+z_1)\,\Gamma(1-\epsilon-z_0+z_1)}{\Gamma(1-2\epsilon-2z_0)}
\end{aligned} \tag{6.48}
$$

Now we introduced a second Mellin Barnes integration to factor the variable x_3. We will not need again to add all the residues due to the fact that we need only certain regions, m_b^0 and $m_b^{-2\epsilon}$ to determine the boundary conditions BC_5^{R} and BC_{11}^{R} respectively. Note that the exponent of the mass scale depends on only one of the Mellin Barnes variable, z_0. This means that the scalings are determined from the residues only due to $\Gamma(-z_0+\cdots)$. The residues due to $\Gamma(-z_1+\cdots)$, on the other hand, should be summed to infinity; which will lead to more complicated functions such as Harmonic Polylogarithms. Upon investigation, we see that we can obtain the scaling $m_b^{-2\epsilon}$ if we integrate over the variable z_0 first. Choosing the integration contour at $-1 < \Re(z_0) < 0$, and closing it to the right, the residues due to $\Gamma(-\epsilon - z_0)$ will result in the $m_b^{-2\epsilon}$ scaling, whereas the residues due to $\Gamma(-z_0)$ will result in the scaling m_b^0. It should be noted that, the residues due to $\Gamma(1-2\epsilon-z_0+z_1)$ and $\Gamma(1-\epsilon-z_0+z_1)$ will not give the desired scalings. After picking up the residues, we are still left with the following integrals:

- Due to $\Gamma(-\epsilon - z_0)$: $m_b^{-2\epsilon}$ scaling:

$$
\begin{aligned}
&-\Gamma(1-\epsilon)\,\Gamma(\epsilon) \\
&\quad \times \int_{-i\infty}^{+i\infty} dz_1 \frac{m_b^{-2\epsilon}\left(e^{i\pi}\right)^{-1-z_1}(1-z)^{z_1}\, z^{-1-z_1}\,\Gamma(-z_1)^2\,\Gamma(1+z_1)^2\,\Gamma(1-\epsilon+z_1)}{\Gamma(2-2\epsilon+z_1)}
\end{aligned} \tag{6.49}
$$

  where we kept terms only up to $\mathcal{O}(m_b^2)$. Now we can perform these remaining integration over z_1. This time we need to sum all the residues, since the exponent of m_b does not contain any Mellin Barnes variable. We will first expand the integral in ϵ, which will make the computation easier. The result we obtain is:

$$
\begin{aligned}
m_b^{-2\epsilon} \Bigg(& \frac{-6H_{0,1} - 3H_0^2 + \pi^2}{6(1-z)\epsilon} \\
& + \frac{1}{6(1-z)} \Big(-12H_0 H_{0,1} + 12H_1 H_{0,1} \\
& \qquad + 18H_{0,0,1} - 24H_{0,1,1} + H_0^3 + 6H_1 H_0^2 - 2\pi^2 H_1 + 6\zeta_3 \Big) \\
& + \frac{\epsilon}{72(1-z)} \Big(288H_1 H_0 H_{0,1} + 144H_0 H_{0,0,1} - 288H_0 H_{0,1,1} - 144H_1^2 H_{0,1} \\
& \qquad + 30\pi^2 H_{0,1} - 432H_1 H_{0,0,1} + 576H_1 H_{0,1,1} - 360H_{0,0,0,1} \\
& \qquad + 576H_{0,0,1,1} - 864H_{0,1,1,1} + 144H_0\zeta_3 - 144H_1\zeta_3 - 3H_0^4 \\
& \qquad - 24H_1 H_0^3 - 72H_1^2 H_0^2 + 3\pi^2 H_0^2 + 24\pi^2 H_1^2 + 7\pi^4 \Big) \\
& + \mathcal{O}(\epsilon^2) \Bigg)
\end{aligned} \tag{6.50}
$$

where we used the shorthand notation for the Harmonic Polylogarithms:

$$
H(\vec{a}; z) = H_{\vec{a}} \tag{6.51}
$$

- Due to $\Gamma(-z_0)$: m_b^0 scaling:

$$
\begin{aligned}
& -\frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)} \int_{-i\infty}^{+i\infty} dz_1 \left(e^{i\pi}\right)^{-1-z_1} (1-z)^{-\epsilon+z_1} z^{-1-z_1} \\
& \times \frac{\Gamma(\epsilon - z_1)\Gamma(-z_1)\Gamma(1+z_1)\Gamma(1-2\epsilon+z_1)\Gamma(1-\epsilon+z_1)}{\Gamma(2-3\epsilon+z_1)}
\end{aligned} \tag{6.52}
$$

Now we will perform these remaining integration over z_1. Similarly, we will expand the integral in ϵ before performing it. The result we get is:

$$
\begin{aligned}
\Big(& \frac{6H_{0,1} + 3H_0^2 - \pi^2}{6(1-z)\epsilon} \\
& + \frac{1}{6(1-z)} \Big(12H_0 H_{0,1} - 12H_1 H_{0,1} - 24H_{0,0,1} + 30H_{0,1,1} \\
& \qquad - 3H_0^3 - 6H_1 H_0^2 + 4\pi^2 H_0 + 2\pi^2 H_1 - 6\zeta_3 \Big) \\
& - \frac{\epsilon}{120(1-z)} \Big(120H_0^2 H_{0,1} + 480H_1 H_0 H_{0,1} + 240H_0 H_{0,0,1}
\end{aligned}
$$

$$
\begin{aligned}
&- 480H_0H_{0,1,1} - 240H_1^2H_{0,1} - 30\pi^2H_{0,1} - 960H_1H_{0,0,1} \\
&+ 1200H_1H_{0,1,1} - 1200H_{0,0,0,1} + 1680H_{0,0,1,1} - 2280H_{0,1,1,1} \\
&+ 240H_0\zeta_3 - 240H_1\zeta_3 - 35H_0^4 - 120H_1H_0^3 - 120H_1^2H_0^2 \\
&+ 125\pi^2H_0^2 + 160\pi^2H_1H_0 + 40\pi^2H_1^2 + 39\pi^4\Big) + \mathcal{O}(\epsilon^2)\Big) \qquad (6.53)
\end{aligned}
$$

The calculation of M_8^{R} is also very similar to M_5^{R} and requires two Mellin Barnes integrations to be introduced. The boundary condition BC_8^R is determined by the $m_b^{-2\epsilon}$ region of this integral. The computation of M_{11}^{R} is straightforward since it includes an extra numerator compared to M_5^{R}, therefore we will skip presenting it here.

6.2.5 $M_9^{\mathbf{R}}$, $M_{10}^{\mathbf{R}}$, $M_{14}^{\mathbf{R}}$, $M_{15}^{\mathbf{R}}$, $M_{16}^{\mathbf{R}}$

All of these masters have four Feynman parameters to start with and will have three Mellin Barnes integrations in the end. $M_{14}^{\mathrm{R}}, M_{15}^{\mathrm{R}}, M_{16}^{\mathrm{R}}$ are the non-planar ones the computations of which will not be displayed here. Here we present M_9^{R} as an example below:

$$
\begin{aligned}
M_9^{\mathrm{R}} =& \,\Gamma(\epsilon+2)\int d\Phi_2 \int_0^1 dx_1\, dx_2\, dx_3\, dx_4\, \delta(1-x_1-x_2-x_3-x_4) \\
&\times (x_1+x_2+x_3+x_4)^{2\epsilon} \\
&\times \left(-x_2\left(s_{23}x_1 + s_{13}(x_1+x_3)\right) - s_{12}x_1(x_2+x_4) + m_b^2(x_1+x_2+x_3+x_4)^2\right)^{-2-\epsilon} \qquad (6.54) \\
=& \,\Gamma(1-\epsilon)\int_{-i\infty}^{+i\infty} dz_0\, dz_1\, dz_2\, \left(e^{i\pi}\right)^{-1-z_1} m_b^{2z_0}\, (1-z)^{-1-\epsilon-z_0+z_1}\, z^{\epsilon+z_0-z_1-z_2} \\
&\times \frac{\Gamma(-z_0)\,\Gamma(-z_1)\,\Gamma(-z_2)\,\Gamma(1+\epsilon+z_0-z_1)\,\Gamma(-2\epsilon-z_0+z_1)}{\Gamma(-2\epsilon-2z_0)\,\Gamma(1-3\epsilon-z_0+z_1)} \\
&\times \Gamma(-\epsilon-z_0+z_1)\,\Gamma(1+\epsilon+z_0-z_2) \\
&\times \Gamma(-\epsilon-z_0+z_2)\,\Gamma(-\epsilon-z_0+z_1+z_2) \qquad (6.55)
\end{aligned}
$$

where z_0, z_1 and z_2 are the newly introduced Mellin Barnes integration variables. We know from the solutions of the differential equations that to determine the boundary condition BC_9^R, we need the scaling $m_b^{-4\epsilon}$. Looking at the gamma functions whose residues will contribute to this scaling we see that the variable z_0, the value of which will give us the actual scaling, and the other Mellin Barnes variables z_1 and z_2 are entangled. This means the residues at specific points for z_1 and z_2 will also contribute to the scaling indirectly. We will see that this is the case in a moment. But first let us simply the situation by carrying out a series of change of variables as the following consecutively:

$$z_1 \to -z_1 \tag{6.56}$$
$$z_2 \to -z_2 + z_0 \tag{6.57}$$
$$z_1 \to -z_0 + z_1 \tag{6.58}$$

With these change of variables we obtain the following expression:

$$\begin{aligned}
&\Gamma(1-\epsilon)\int_{-i\infty}^{+i\infty} dz_0\, dz_1\, dz_2\, \left(e^{i\pi}\right)^{-1-z_0+z_1} m_b^{2z_0}\, (1-z)^{-1-\epsilon-z_1}\, z^{\epsilon-z_0+z_1+z_2} \\
&\times \frac{\Gamma(-z_0)\,\Gamma(-z_0+z_1)\,\Gamma(-z_0+z_2)\,\Gamma(-2\epsilon-z_1)}{\Gamma(1-z_1-3\epsilon)\,\Gamma(-2z_0-2\epsilon)} \\
&\times \Gamma(-\epsilon-z_2)\,\Gamma(-\epsilon-z_1)\,\Gamma(1+\epsilon+z_1)\,\Gamma(1+\epsilon+z_2) \\
&\times \Gamma(z_0-z_1-z_2-\epsilon)
\end{aligned} \tag{6.59}$$

Although not much simpler, this expression is still easier to work with. The integration contours are also shifted as $0 < \Re(z_0) < 1$, $-1 < \Re(z_1) < 0$ and $0 < \Re(z_2) < 1$, and now we can determine the residues that will contribute to the scaling $m_b^{-4\epsilon}$. $\Gamma(-z_0)$ cannot give any residues that would lead to this scaling, therefore we start with $\Gamma(-z_0+z_1)$:

- Taking the series of residues at points $z_0 = z_1 + n_0$ due to $\Gamma(-z_0+z_1)$, where n_0 is an integer we get:

$$\begin{aligned}
-\sum_{n_0}^{\infty} &(-1)^{1-n_0} \left(e^{i\pi}\right)^{-1-n_0} \Gamma(1-\epsilon) \int_{-i\infty}^{+i\infty} dz_1\, dz_2\, m_b^{2(n_0+z_1)}\, (1-z)^{-1-\epsilon-z_1}\, z^{\epsilon-n_0+z_2} \\
&\times \frac{\Gamma(-n_0-z_1)\,\Gamma(-n_0-z_1+z_2)\,\Gamma(-z_1-2\epsilon)\,\Gamma(-\epsilon-z_1)}{n_0!\,\Gamma(1-z_1-3\epsilon)\,\Gamma(-2(n_0+z_1+\epsilon))} \\
&\times \Gamma(-\epsilon-z_2)\,\Gamma(n_0-z_2-\epsilon)\,\Gamma(1+z_1+\epsilon)\,\Gamma(1+z_2+\epsilon)
\end{aligned} \tag{6.60}$$

  Looking at the exponent of the mass scale, i.e. $m_b^{2(n_0+z_1)}$, we see that only the residues due to $\Gamma(-z_1-2\epsilon)$ can result in the right scaling. Taking also these residues at points $z_1 = n_1 - 2\epsilon$, where n_1 is an integer, we perform the integration over z_1 as well:

$$\sum_{n_0}^{\infty}\sum_{n_1}^{\infty}(-1)^{2-n_0-n_1}\left(e^{i\pi}\right)^{-1-n_0} m_b^{2(n_0+n_1-2\epsilon)}\,(1-z)^{-1+\epsilon-n_1}$$
$$\times\Gamma(1-\epsilon)\int_{-i\infty}^{+i\infty}dz_2\,z^{\epsilon-n_0+z_2}$$
$$\times\frac{\Gamma(1-\epsilon+n_1)\Gamma(-z_2-\epsilon)\Gamma(n_0-z_2-\epsilon)\Gamma(-n_1+\epsilon)}{n_0!\,n_1!\,\Gamma(1-n_1-\epsilon)\Gamma(-2(n_0+n_1-\epsilon))}$$
$$\times\Gamma(1+z_2+\epsilon)\Gamma(-n_0-n_1+2\epsilon)\Gamma(-n_0-n_1+z_2+2\epsilon)\qquad(6.61)$$

Now the mass scale becomes $m_b^{2(n_0+n_1-2\epsilon)}$, which means for values $n_0=0$, $n_1=0$ we obtain the right scaling at zeroth order. Therefore we do not need to carry out the summations but just need to perform the integration over z_2. Setting $n_0=0$, $n_1=0$ we obtain:

$$\Gamma(1-\epsilon)\Gamma(\epsilon)\int_{-i\infty}^{+i\infty}dz_2\,m_b^{-4\epsilon}\,e^{-i\pi}\,(1-z)^{-1+\epsilon}\,z^{z_2+\epsilon}$$
$$\times\Gamma(-z_2-\epsilon)^2\,\Gamma(1+z_2+\epsilon)\Gamma(z_2+2\epsilon)\qquad(6.62)$$

We expand in ϵ to perform this integral. We obtain:

$$m_b^{-4\epsilon}\Bigg(\frac{-6H_{0,1}+3H_0^2+6H_1H_0+2\pi^2}{6(1-z)\epsilon}$$
$$+\frac{\epsilon}{360(1-z)}\Big(-90\pi^2H_{0,1}-360H_{0,1,1,1}+15H_0^4+60H_1H_0^3+90H_1^2H_0^2$$
$$+75\pi^2H_0^2+60H_1^3H_0+150\pi^2H_1H_0+30\pi^2H_1^2+41\pi^4\Big)$$
$$-\frac{-6H_{0,1,1}+H_0^3+3H_1H_0^2+3H_1^2H_0+\pi^2H_0+\pi^2H_1+6\zeta_3}{6(1-z)}+\mathcal{O}(\epsilon^2)\Bigg)$$
$$(6.63)$$

- The other contribution is from the residues due to $\Gamma(-z_0+z_2)$. Taking the residues at $z_0=n_0+z_2$ we have:

$$-\sum_{n_0}^{\infty}(-1)^{1-n_0}\Gamma(1-\epsilon)\int_{-i\infty}^{+i\infty}dz_1\,dz_2\,\left(e^{i\pi}\right)^{-1-n_0+z_1-z_2} m_b^{2(n_0+z_2)}\,(1-z)^{-1-\epsilon-z_1}$$

$$
z^{\epsilon-n_0+z_1}\,\frac{\Gamma\left(-n_0-z_2\right)\Gamma\left(-n_0+z_1-z_2\right)\Gamma\left(-z_1-2\epsilon\right)\Gamma\left(-z_1-\epsilon\right)}{n_0!\,\Gamma\left(1-z_1-3\epsilon\right)\Gamma\left(-2(n_0+z_2+\epsilon)\right)}
\times\Gamma\left(n_0-z_1-\epsilon\right)\Gamma\left(-z_2-\epsilon\right)\Gamma\left(1+z_1+\epsilon\right)\Gamma\left(1+z_2+\epsilon\right) \tag{6.64}
$$

Now looking at the expression we see that if we first take the residues due to $\Gamma\left(-n_0+z_1-z_2\right)$ at points $z_2=n_1-n_0+z_1$ and then due to $\Gamma\left(-z_1-2\epsilon\right)$ at points $z_1=n_2-2\epsilon$ we will obtain the right scaling:

$$
\sum_{n_0}^{\infty}\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}(-1)^{2-n_0-n_1}\left(e^{i\pi}\right)^{-1-n_1}\Gamma\left(1-\epsilon\right)\int_{-i\infty}^{+i\infty}dz_1\,m_b^{2(n_1+z_1)}\left(1-z\right)^{-1+\epsilon-z_1}
\times z^{-n_0+z_1+\epsilon}\frac{\Gamma\left(-n_1-z_1\right)\Gamma\left(-z_1-2\epsilon\right)\Gamma\left(-z_1-\epsilon\right)\Gamma\left(n_0-z_1-\epsilon\right)}{n_0!\,n_1!\,\Gamma\left(1-z_1-3\epsilon\right)\Gamma\left(-2(n_1+z_1+\epsilon)\right)}
\times\Gamma\left(n_0-n_1-z_1-\epsilon\right)\Gamma\left(1+z_1+\epsilon\right)\Gamma\left(1-n_0+n_1+z_1+\epsilon\right) \tag{6.65}
$$

$$
=-\sum_{n_0}^{\infty}\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}(-1)^{3-n_0-n_1-n_2}\left(e^{i\pi}\right)^{-1-n_1}\Gamma\left(1-\epsilon\right)m_b^{2(n_1+n_2-2\epsilon)}
\times\left(1-z\right)^{-1-n_2+\epsilon}z^{-n_0+n_2-\epsilon}
\times\frac{\Gamma\left(1+n_2-\epsilon\right)\Gamma\left(1-n_0+n_1+n_2-\epsilon\right)\Gamma\left(-n_2+\epsilon\right)}{n_0!\,n_1!\,n_2!\,\Gamma\left(1-n_2-\epsilon\right)\Gamma\left(-2(n_1+n_2-\epsilon)\right)}
\times\Gamma\left(n_0-n_2+\epsilon\right)\Gamma\left(n_0-n_1-n_2+\epsilon\right)\Gamma\left(-n_1-n_2+2\epsilon\right) \tag{6.66}
$$

Now we set $n_1=0$ and $n_2=0$ to obtain:

$$
-\sum_{n_0}^{\infty}(-1)^{3-n_0}\,e^{-i\pi}\,m_b^{-4\epsilon}\left(1-z\right)^{-1+\epsilon}z^{-n_0-\epsilon}\,\Gamma\left(1-\epsilon\right)\Gamma\left(\epsilon\right)
\times\frac{\Gamma\left(1-n_0-\epsilon\right)\Gamma\left(n_0+\epsilon\right)^2}{n_0!} \tag{6.67}
$$

$$
=-\frac{m_b^{-4\epsilon}\left(e^{i\pi}\right)^{-2-\epsilon}\Gamma\left(1-\epsilon\right)^2\Gamma\left(\epsilon\right)^3}{1-z} \tag{6.68}
$$

To arrive at the final result we add these two contributions, which we will skip displaying since our purpose here is only to show the procedure. The boundary condition BC_9^R is presented in the Appendix. As seen from the arguments above, the way to handle the multiple Mellin Barnes integrations is quite tedious such that it is not feasible to perform the integrations by hand following the reasoning explained above. That is why we use the Mathematica package MB [131, 132], which implements this method algorithmically, to resolve the singularities and perform the Mellin Barnes integrations.

The calculation of the specific regions of the master integrals $M_{10}^{\mathrm{R}}, M_{14}^{\mathrm{R}}, M_{15}^{\mathrm{R}}, M_{16}^{\mathrm{R}}$ follow similarly to the one that discussed above, so we do not discuss these here.

6.2.6 Soft Limits

The master integrals in the soft limit where $z \to 1$ are computed separately. Since these are more easier than the calculation of master integrals themselves, we do not describe them here.

However, we would like to briefly discuss how to proceed with the expansion in z. In order to obtain the soft limits, we need to perform two expansions in our case: around the vanishing bottom quark mass, and the soft expansion. From the first look, it is not clear which expansion to perform first. The reason is that to obtain the hadronic cross section an integration over z should be performed; and in principle, the expansion in m_b^2 can be carried out after the z integration. Then, expansion in z should be carried out before the expansion in m_b^2. Doing so creates scalings with a different structure in the vanishing bottom mass for pieces of the cross section involving particular master integrals, i.e. $\left(m_b^2\right)^{-3\epsilon}$. However, these pieces, when added together, cancel. Therefore we can safely perform the expansion in vanishing bottom quark mass first.

Chapter 7

Double Virtual Contribution

In this chapter we will present the results for the double-virtual contribution. As in the previous chapter, we will first give a brief description on how we calculated the matrix elements. Then we will present the master integrals as Feynman diagrams with the subsequent boundary conditions given in the Appendix.

The procedure we follow to perform the computation is similar to that of real-virtual, with an exception that we do not use reverse unitarity method. The phase space integral is over one particle only, Higgs, therefore the computation is rather straighforward. We generate the Feynman diagrams viaFEYNARTS [129]. And use our own MATHEMATICA code to dress the diagrams, i.e. assign momenta, spin and colour indices, as well as applying the Feynman rules and contracting the indices. The spinor/colour traces are evaluated via FEYNCALC [130]. In this case there are no ghosts.

7.1 Matrix Elements

In this section we present our results for the double virtual contribution to the unrenormalized squared matrix element for the top-bottom interference for the process $gg \to H$ in terms of master integrals. The amplitude consists of two pieces. The first piece is the leading order of this process in the full theory with bottom quarks running in the loop interfered with one loop correction to the ggH vertex in the effective theory, while the second piece is the two-loop correction to the to the same process in the full theory interfered with the Higgs-gluon-gluon vertex in the effective theory.

To express our results we use the dimensionless variable:

$$x = \frac{\sqrt{1 - 4m_b^2/s} - 1}{\sqrt{1 - 4m_b^2/s} + 1} \tag{7.1}$$

so that for $m_b^2 \to 0$, we have $x = -m_b^2/s + \mathcal{O}\left(m_b^4\right) \to 0$. With this choice, the dimension of the masters would factor from their analytic expressions as $(-s)^{2d-\nu}$, where ν is the total inverse power of propagators. Therefore we conveniently set s to -1 to obtain an expression in x. The dimension of the masters can easily be recovered from dimensional analysis as just mentioned.

We should note that for generic values of fermion mass running in the loop, it is necessary to use a variable such as x in order to distinguish the kinematical regions and properly perform the analytic continuation from one region to another. Here by setting $s = -1$ we are in the space-like region. However in our case $4m_b^2 < s$, therefore we are above the threshold, where x is real valued. Therefore we have to analytically continue our results from the space-like region where $s < 0$ to the physical region where $s > 0$. In both cases x is positive, therefore the analytic continuation is trivial:

$$-\frac{m_b^2}{s} \to \frac{m_b^2 - i0}{s + i0} = -\frac{m_b^2}{s} + i0, \quad \text{with} \quad s > 0 \tag{7.2}$$

Now let us turn to the amplitude. As explained in Chapter 5, the the double virtual contribution to the cross section is given by:

$$\sigma_{ij\to H}^{\text{V;int.}} = \frac{n_{ij}}{2s} \int \mathrm{d}\Phi_1(p_1, p_2; p_H)\; 2\Re\left\{\mathcal{A}_{ij\to H}^{(0)} \left(\mathcal{B}_{ij\to H}^{(1)}\right)^* + \mathcal{A}_{ij\to H}^{(1)} \left(\mathcal{B}_{ij\to H}^{(0)}\right)^*\right\} \tag{7.3}$$

where $\mathcal{A}^{(n)}$ and $\mathcal{B}^{(n)}$ are the nth corrections to the matrix element for the process $gg \to H$ in the full theory with bottom quarks running in the loop and in the effective theory respectively. Here we will describe how we computed these matrix elements.

The first piece in Equation 7.3, the leading order for the production of a Higgs boson via gluon fusion in the full theory interfered with the NLO matrix element in the effective theory is then given by:

$$\sum_{\text{pols,cols}} 2\Re\,\mathcal{A}^{(0)}_{gg\to H}\left(\mathcal{B}^{(1)}_{gg\to H}\right)^{*} = \frac{1}{v}\,c_H g_s^4\,\frac{4\left(N_C^2-1\right)}{(1-\epsilon)\,s^2}A_{\text{LO}}\,B^{*}_{\text{NLO}}, \tag{7.4}$$

where we explicitly wrote the summation over the polarizations and colors of the gluons, g_s is the strong coupling, v is the vacuum expectation value of the Higgs field, c_H is the Wilson coefficient, s is the center of mass energy, N_C is the color factor of SU(3). Here, we used the formulation we introduced in Chapter 4.5 to obtain the interference of the two amplitudes at the right hand side of Equation 7.4. A_{LO} is given by

$$A_{\text{LO}} = \frac{2x\left(x^2(\epsilon-1)-2x(\epsilon+1)+\epsilon-1\right)}{(x-1)^4}M_3^{\text{B}}(\epsilon)+\frac{4x\epsilon}{(x-1)^2}M_2^{\text{B}}(\epsilon) \tag{7.5}$$

and where B_{NLO} is:

$$B_{\text{NLO}} = C_A\left(-\frac{(4-2\epsilon)^3-16(4-2\epsilon)^2+68(4-2\epsilon)-88}{8\epsilon}\right)M_0^{\text{B}}(\epsilon) \tag{7.6}$$

where $M_2^{\text{B}}(\epsilon)$ and $M_3^{\text{B}}(\epsilon)$ are massive bubble and triangle master integrals respectively, that are given in equations 4.53 and 4.52; whereas $M_0^{\text{B}}(\epsilon)$ is the one loop massless bubble master, that is given in Equation 3.68. The results we just presented for A_{LO} and B_{NLO} are in agreement with the literature [133].

The second piece in Equation 7.3, the double virtual amplitude for the production of Higgs boson via gluon fusion in the full theory interfered with the LO matrix element in the effective theory is:

$$\sum_{\text{pols,cols}} 2\Re\,\mathcal{A}^{(1)}_{gg\to H}\left(\mathcal{B}^{(0)}_{gg\to H}\right)^{*} = \frac{1}{v}\,c_H g_s^4\,\frac{4\left(N_C^2-1\right)}{(1-\epsilon)\,s^2}A_{\text{V}}. \tag{7.7}$$

We present A_{V} in Appendix in terms of master integrals. For convenience we have set $s=1$.

7.2 Master Integrals

The first piece, where we have LO process with bottom quarks running in the loop interfered with one loop correction to the effective process, results in factorizable one loop integrals, therefore trivial. For the second piece we find three distinct topologies

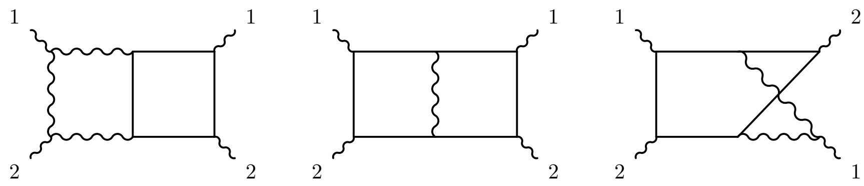


Figure 7.1: Topologies $T_i^{\mathrm{V}}[\nu_1, \ldots, \nu_7]$, $i = 1, 2, 3$, appearing in the computation of the virtual contributions. Wavy lines indicate massless internal propagators and external legs, while plain lines indicate massive internal propagators and external legs.

that we will denote as $T_1^{\mathrm{V}}[\nu_1, \ldots, \nu_7]$, $T_2^{\mathrm{V}}[\nu_1, \ldots, \nu_7]$ and $T_3^{\mathrm{V}}[\nu_1, \ldots, \nu_7]$, which are depicted in Figure 7.1. We define these as the following:

$$T_1^{\mathrm{V}}[\nu_1, \ldots, \nu_5] = \mathrm{e}^{2\epsilon\gamma_E} \int \frac{d^d k}{i\pi^{d/2}} \frac{d^d l}{i\pi^{d/2}} \frac{1}{D_{11}^{\nu_1} D_{12}^{\nu_2} D_{13}^{\nu_3} D_{14}^{\nu_4} D_{15}^{\nu_5} D_{16}^{\nu_6} D_{17}^{\nu_7}} \tag{7.8}$$

where

$$\begin{aligned}
D_{11} &= k^2 & D_{15} &= (l + p_1 + p_2)^2 - m_b^2 \\
D_{12} &= (k + p_1)^2 & D_{16} &= (l + p_1)^2 - m_b^2 \\
D_{13} &= (k + p_1 + p_2)^2 & D_{17} &= l^2 - m_b^2 \\
D_{14} &= (k - l)^2 - m_b^2
\end{aligned} \tag{7.9}$$

$$T_2^{\mathrm{V}}[\nu_1, \ldots, \nu_5] = \mathrm{e}^{2\epsilon\gamma_E} \int \frac{d^d k}{i\pi^{d/2}} \frac{d^d l}{i\pi^{d/2}} \frac{1}{D_{21}^{\nu_1} D_{22}^{\nu_2} D_{23}^{\nu_3} D_{24}^{\nu_4} D_{25}^{\nu_5} D_{26}^{\nu_6} D_{27}^{\nu_7}} \tag{7.10}$$

where

$$\begin{aligned}
D_{21} &= k^2 - m_b^2 & D_{25} &= (l + p_1 + p_2)^2 - m_b^2 \\
D_{22} &= (k + p_2)^2 - m_b^2 & D_{26} &= (l + p_2)^2 - m_b^2 \\
D_{23} &= (k + p_1 + p_2)^2 - m_b^2 & D_{27} &= l^2 - m_b^2 \\
D_{24} &= (k - l)^2
\end{aligned} \tag{7.11}$$

$$T_3^{\mathrm{V}}[\nu_1, \ldots, \nu_5] = \mathrm{e}^{2\epsilon\gamma_E} \int \frac{d^d k}{i\pi^{d/2}} \frac{d^d l}{i\pi^{d/2}} \frac{1}{D_{21}^{\nu_1} D_{22}^{\nu_2} D_{23}^{\nu_3} D_{24}^{\nu_4} D_{25}^{\nu_5} D_{36}^{\nu_6} D_{37}^{\nu_7}} \tag{7.12}$$

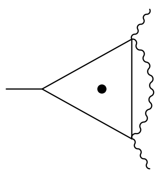
where

$$\begin{aligned}
D_{31} &= k^2 - m_b^2 & D_{35} &= (l + p_1)^2 - m_b^2
\end{aligned}$$

$$\begin{aligned} D_{32} &= (k-l-p_1)^2 - m_b^2 & D_{36} &= l^2 - m_b^2 \\ D_{33} &= (k+p_1+p_2)^2 - m_b^2 & D_{37} &= (k-l)^2 \\ D_{34} &= (l+p_1+p_2)^2 - m_b^2 & & \end{aligned} \tag{7.13}$$

where k and l are the loop momenta. After the reduction, we find 17 master integrals. We follow reference [14] and choose for our master integrals the following:

where we omitted master M_{12}^{V} which has a non-trivial numerator and can be written as

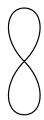


$$= \frac{1}{2}\left(T_2^{\mathrm{V}}[0,1,0,1,1,-1,1] - T_2^{\mathrm{V}}[0,0,0,1,1,0,1]\right) = M_{12}^{\mathrm{V}}.$$

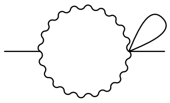
Some of these master integrals can be expressed in more than one topology, however for convenience we only indicated one of them.

We obtain the differential equations with respect to the parameter x, as an expansion around $x = 0$, and perform the reduction as usual. Then we find the solutions to the differential equations using the method described in Section 3.3.3. Now we will describe how we calculated the needed regions for the master integrals. Here in the double-virtual case, we extracted the regions with the program Asy [110, 111] that we briefly mentioned in Section 3.4 on Expansion by Regions, as opposed to real-virtual case where we directly used Mellin Barnes representation method. However, some of the integrals are calculated in both ways and cross checked. We present all boundary conditions as an expansion in ϵ in Appendix C.1. Following ref. [41], the solutions to the differential equations, i.e. the master integrals written in terms of boundary conditions, can be found in Appendix B.1, together with the analytic expressions for masters, which are given in Appendix D.1 for completeness.

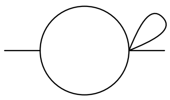
Below we will present explicit computation of some of the regions for integrals M_8^{V} and M_{15}^{V}.



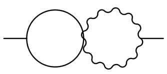
$$= T_1^{\mathrm{V}}[0,0,0,1,0,1,0] = M_1^{\mathrm{V}},$$



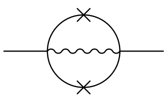
$$= T_1^{\mathrm{V}}[1,0,1,1,0,0,0] = M_2^{\mathrm{V}},$$



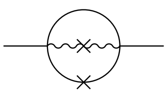
$$= T_1^{\mathrm{V}}[0,0,0,1,1,0,1] = M_3^{\mathrm{V}},$$



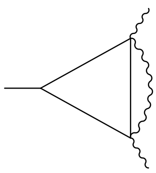
$$= T_1^{\mathrm{V}}[1,0,1,0,1,0,1] = M_4^{\mathrm{V}},$$



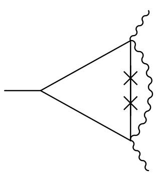
$$= T_1^{\mathrm{V}}[1,0,0,2,2,0,0] = M_5^{\mathrm{V}},$$



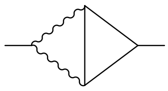
$$= T_1^{\mathrm{V}}[2,0,0,2,1,0,0] = M_6^{\mathrm{V}},$$

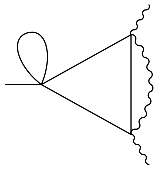


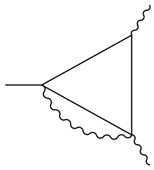
$$= T_1^{\mathrm{V}}[0,1,0,1,1,0,1] = M_7^{\mathrm{V}},$$

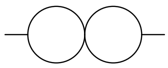


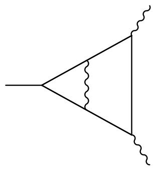
$$= T_1^{\mathrm{V}}[0,1,0,3,1,0,1] = M_8^{\mathrm{V}},$$

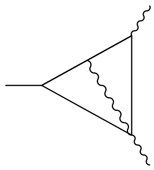
 $= T_1^{\mathrm{V}}[1,0,1,1,1,0,1] = M_9^{\mathrm{V}},$

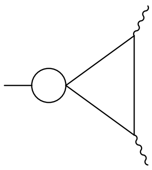
 $= T_1^{\mathrm{V}}[0,0,0,1,1,1,1] = M_{10}^{\mathrm{V}},$

 $= T_1^{\mathrm{V}}[0,0,1,1,0,1,1] = M_{11}^{\mathrm{V}},$

 $= T_2^{\mathrm{V}}[1,0,1,0,1,0,1] = M_{13}^{\mathrm{V}},$

 $= T_2^{\mathrm{V}}[1,0,1,1,1,1,1] = M_{14}^{\mathrm{V}},$

 $= T_2^{\mathrm{V}}[0,1,1,1,1,0,1] = M_{15}^{\mathrm{V}},$

 $= T_2^{\mathrm{V}}[1,0,1,0,1,1,1] = M_{16}^{\mathrm{V}},$

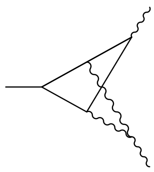
 $= T_3^{\mathrm{V}}[1,0,1,1,1,1,1] = M_{17}^{\mathrm{V}},$

Figure 7.2: Master Integrals that Appear in Double-Virtual Contribution

7.2.1 $M_8^{\mathbf{V}}$

We are interested in the $x^{-1-2\epsilon}$ scaling of this integral. This piece will help us determine the BC_8^{V}. $x^{-1-2\epsilon}$ scaling of M_8^{V} consists of two different regions:

$$M_{8,2\epsilon}^{\mathrm{V}} = x^{-1-2\epsilon}\, e^{2\epsilon\,\gamma_E} \left(M_{8-1,2\epsilon}^{\mathrm{V}} + M_{8-2,2\epsilon}^{\mathrm{V}} \right) \tag{7.14}$$

Here, we will present the computation of $M_{8-1,2\epsilon}^{\mathrm{V}}$, and present the result for $M_{8-2,2\epsilon}^{\mathrm{V}}$, together with the final analytic expression for $M_{8,2\epsilon}^{\mathrm{V}}$. First, let us start with an expression in Feynman parameters representation where the loop integration is already taken care of. We also introduce an additional regulator, δ, to the propagators of this integral in the momentum representation. Recall that we did this also for the computation of the regions for the massive triangle integral in Section 3.4, which can be seen in Equation 3.167. The reason for the need to have this regulator is that when the integral is split into regions, spurious singularities are created which need to be regularized. In the final results for the regions, when they are expanded in the regulator, these singularities will appear as poles in δ, for $\delta \to 0$. When the regions are added together, we expect that the singularities cancel. This will indeed be the case as we will see at the end of this computation.

Now let us start with the Feynman parameterized expression for $M_{8-1,2\epsilon}^{\mathrm{V}}$:

$$\begin{aligned} M_{8-1,2\epsilon}^{\mathrm{V}} = &- x^{\delta} \int_0^1 dx_1 dx_2 dx_3 dx_4\, \delta\left(1 - x_1 - x_2 - x_3 - x_4\right) \\ &\times \frac{\Gamma\left(2+2\epsilon\right)}{\Gamma\left(1+\delta\right)\Gamma\left(1-\delta\right)}\, x_2^{-\delta}\, x_3^{\delta}\, x_4^{2} \left(x_1 x_2 + x_1 x_4 + x_2 x_4\right)^{3\epsilon} \\ &\times \left(x_1 x_2^2 + x_1 x_2 x_3 + 2 x_1 x_2 x_4 + x_2^2 x_4 + x_2 x_3 x_4 + x_1 x_4^2 + x_2 x_4^2 \right) \Big)^{-2-2\epsilon} \end{aligned} \tag{7.15}$$

$$\begin{aligned} = &- x^{\delta}\, \frac{\Gamma\left(1+2\epsilon-\delta\right)}{\Gamma\left(1-\delta\right)} \int_0^1 dx_1 dx_2\, x_2^{-1-2\delta}\, \left(1+x_1\right)^{-1-\delta} \left(1+x_2\right)^{-1-2\epsilon+\delta} \\ &\times \left(x_2 + x_1(1+x_2)\right)^{-1+\epsilon+\delta} \end{aligned} \tag{7.16}$$

where in the second line, we set $x_4 = 1$ and integrated over x_3 using Equation 3.74. Now we can introduce a Mellin Barnes integral to split the sum in the term $(x_2 + x_1(1+x_2))^{-1+\epsilon+\delta}$ and factor x_1 parameter. With this we also subsequently perform the integration over x_1, and then x_2 to obtain:

$$
\begin{aligned}
&- x^{\delta} \frac{\Gamma(2+\epsilon)\Gamma(1+2\epsilon-\delta)}{\Gamma(1-\delta)\Gamma(1+\delta)\Gamma(1-\epsilon-\delta)} \\
&\int_{-i\infty}^{+i\infty} dz_1 \frac{\Gamma(-z_1)\Gamma(1-\epsilon+z_1)\Gamma(z_1-2\delta)\Gamma(1-\epsilon+z_1-\delta)\Gamma(\epsilon-z_1+\delta)}{\Gamma(2+\epsilon+z_1-2\delta)} \qquad (7.17)
\end{aligned}
$$

We choose the integration contour at $0 < \Re(z_1) < 1$ and pick the residues to the right. Before picking up the residues, we expand the integral around $\delta \to 0$ up to $\mathcal{O}(\delta)$. We obtain:

$$
\begin{aligned}
&-\epsilon^2 \Gamma(-\epsilon)\Gamma(\epsilon)\Gamma(2\epsilon)\left(\frac{1}{\delta} + 2\gamma_E + \log(x) + 4\psi^{(0)}(1-\epsilon) - \psi^{(0)}(\epsilon) - \psi^{(0)}(1+2\epsilon)\right) \\
&+\frac{\Gamma(\epsilon)}{1+2\epsilon}\Big((1+\epsilon)\Gamma(\epsilon) + 2\epsilon^2(1+2\epsilon)\Gamma(-\epsilon)\Gamma(2\epsilon) \\
&\qquad\qquad \times \left(2\psi^{(0)}(1-\epsilon) - \psi^{(0)}(\epsilon) - \psi^{(0)}(2+\epsilon)\right)\Big) \\
&-\frac{\Gamma(2+\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-\epsilon)} \int_{-i\infty}^{+i\infty} dz_1 \frac{\Gamma(\epsilon-z_1)\Gamma(-z_1)\Gamma(z_1)\Gamma(1-\epsilon+z_1)^2}{\Gamma(2+\epsilon+z_1)} \qquad (7.18)
\end{aligned}
$$

Now we perform the Mellin Barnes integration. We pick up the residues:

- Due to $\Gamma(-z_1)$:

$$
\begin{aligned}
&\frac{\Gamma(2+\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-\epsilon)} \sum_{n_1=1}^{\infty} \frac{(-1)^{1-n_1}\Gamma(\epsilon-n_1)\Gamma(n_1)\Gamma(1-\epsilon+n_1)^2}{n_1!\,\Gamma(2+\epsilon+n_1)} \\
&= -\frac{(2+\epsilon)\Gamma(2-\epsilon)^2\Gamma(-1+\epsilon)\Gamma(2+\epsilon)\Gamma(1+2\epsilon)\left(\psi^{(0)}(2+\epsilon) - \psi^{(0)}(1+2\epsilon)\right)}{(-1+\epsilon)\Gamma(1-\epsilon)\Gamma(3+\epsilon)} \qquad (7.19)
\end{aligned}
$$

- Due to $\Gamma(\epsilon - z_1)$:

$$
\begin{aligned}
&\frac{\Gamma(2+\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-\epsilon)} \sum_{n_1=1}^{\infty} \frac{(-1)^{1-n_1}\Gamma(-\epsilon-n_1)\Gamma(\epsilon+n_1)\Gamma(1+n_1)^2}{n_1!\,\Gamma(2+2\epsilon+n_1)} \\
&= -\frac{\epsilon(1+\epsilon)\Gamma(-1-\epsilon)\Gamma(\epsilon)^2}{(1+2\epsilon)(2+2\epsilon)\Gamma(-\epsilon)}\, {}_3F_2(\{1,2,1+\epsilon\},\{2+\epsilon,3+2\epsilon\},1) \qquad (7.20)
\end{aligned}
$$

where ${}_3F_2(\{1,2,1+\epsilon\},\{2+\epsilon,3+2\epsilon\},1)$ is the Hypergeometric function. Now combining everything and expanding in ϵ we get:

$$e^{2\epsilon\,\gamma_E}\,M^{\mathrm{V}}_{8-1,2\epsilon}=\left(\frac{1}{\delta}\left(\frac{1}{2\epsilon}+\epsilon\,\frac{\pi^2}{4}\right)+\frac{1}{2\epsilon^2}+\frac{\log(x)}{2\epsilon}+\frac{\epsilon}{2}\left(\frac{\pi^2\log(x)}{2}+\frac{19\zeta_3}{3}\right)+\mathcal{O}(\epsilon^2)\right)\tag{7.21}$$

As we can see, this result has a pole in the additional regulator that we introduced, δ. This singularity should cancel when all terms corresponding to a specific scaling are added together. Indeed this is the case. We give the result for the second part to this scaling, $M^{\mathrm{V}}_{8-2,2\epsilon}$:

$$e^{2\epsilon\,\gamma_E}\,M^{\mathrm{V}}_{8-2,2\epsilon}=\left(\frac{1}{\delta}\left(-\frac{1}{2\epsilon}-\epsilon\,\frac{\pi^2}{4}\right)+\frac{1}{2\epsilon^2}+\frac{\log(x)}{2\epsilon}+\frac{\epsilon}{2}\left(\frac{\pi^2\log(x)}{2}+\frac{19\zeta_3}{3}\right)+\mathcal{O}(\epsilon^2)\right)\tag{7.22}$$

Adding these together, we obtain the final result:

$$M^{\mathrm{V}}_{8,2\epsilon}=x^{-1-2\epsilon}\left(\frac{1}{\epsilon^2}+\frac{\log(x)}{\epsilon}+\frac{\epsilon}{2}\left(\pi^2\log(x)+\frac{19\zeta_3}{3}\right)+\mathcal{O}(\epsilon^2)\right)\tag{7.23}$$

7.2.2 $M^{\mathbf{V}}_{15}$

Here in this section, we will present the computation of the $x^{-2\epsilon}$ scaling of the master integral M^{V}_{15}. This is a five propagators integral, and in Feynman parameters, this integral is given as:

$$\begin{aligned}M^{\mathrm{V}}_{15,2\epsilon}=&\,e^{2\epsilon\,\gamma_E}\,x^{-2\epsilon}\,\Gamma\left(1+2\epsilon\right)\int_0^1 dx_1dx_2dx_3dx_4dx_5\\&\delta\left(1-x_1-x_2-x_3-x_4-x_5\right)\left(x_1x_3+x_2x_3+x_1x_5+x_2x_5+x_3x_5\right)^{-1+3\epsilon}\\&\times\Big(x_1^2x_3+2x_1x_2x_3+x_2^2x_3+x_1x_3^2+x_2x_3^2+x_1x_3x_4+x_2x_3x_4+x_1^2x_5\\&+2x_1x_2x_5+x_2^2x_5+2x_1x_3x_5+2x_2x_3x_5+x_3^2x_5+x_2x_4x_5+x_3x_4x_5\Big)^{-1-2\epsilon}\end{aligned}\tag{7.24}$$

where $x_i, i=1\cdots5$ are the Feynman parameters. The procedure to calculate this integral will be the following: we will change the boundaries of the integral according to Cheng-Wu theorem, integrate over some of the variables according to equation 3.74 and introduce two Mellin Barnes representations to turn a sum into a product as depicted in Equation 3.75. As the first step let us set one of the Feynman parameters to one and change the boundaries of the rest of the integrations to from zero to one to from one to infinity as the Cheng-Wu theorem suggests:

$$e^{2\epsilon\,\gamma_E}\,x^{-2\epsilon}\,\Gamma\left(1+2\epsilon\right)\int_0^\infty dx_1dx_2dx_4dx_5\Big(x_5+\left(x_1+x_2\right)\left(1+x_5\right)\Big)^{-1+3\epsilon}$$
$$\times\Bigg(x_1+x_5+2x_1x_5+\left(x_1+x_2\right)^2\left(1+x_5\right)+x_2\left(1+x_5\right)+x_2x_5$$
$$+x_4\Big(x_1+x_5x_2\left(1+x_5\right)\Big)\Bigg)^{-1-2\epsilon} \tag{7.25}$$

where we set $x_3=1$. The choice of which parameter to set to one at the beginning is intuitive, one usually tries out a couple of different ones separately and pick the one that makes the rest of the calculation easier. In this case we choose this parameter to be x_3, which after some simplification allows the parameter x_4 to be integrated easily since Equation 7.25 has the form of Equation 3.74 for the variable x_4. Integrating over x_4 we get:

$$e^{2\epsilon\,\gamma_E}\,x^{-2\epsilon}\,\Gamma\left(2\epsilon\right)\int_0^\infty dx_1dx_2dx_5\frac{\left(1+x_1+x_2\right)^{-2\epsilon}\Big(x_1+x_2+\left(1+x_1+x_2\right)x_5\Big)^{-1+\epsilon}}{x_1+x_2+x_5\left(1+x_2\right)} \tag{7.26}$$

Now we can introduce the Mellin Barnes representation to turn the sum containing the variable x_5 into a product:

$$e^{2\epsilon\,\gamma_E}\,x^{-2\epsilon}\,\frac{\Gamma\left(2\epsilon\right)}{\Gamma\left(1-\epsilon\right)}\int_{-i\infty}^{+i\infty}dz_1\int_0^\infty dx_1dx_2dx_5$$
$$\times\frac{\Gamma\left(-z_1\right)\Gamma\left(1-\epsilon+z_1\right)\left(x_1+x_2\right)^{z_1}\left(1+x_1+x_2\right)^{-1-\epsilon-z_1}x_5^{-1+\epsilon-z_1}}{\left(x_1+x_2+x_5\left(1+x_2\right)\right)} \tag{7.27}$$

where z_1 is the new Mellin Barnes variable. This expression is now of the form of Equation 3.74 in variable x_5 so we can integrate it out. Then we are left with:

$$e^{2\epsilon\,\gamma_E}\,x^{-2\epsilon}\,\frac{\Gamma\left(2\epsilon\right)}{\Gamma\left(1-\epsilon\right)}\int_{-i\infty}^{+i\infty}dz_1\Gamma\left(\epsilon-z_1\right)\Gamma\left(-z_1\right)$$
$$\Gamma\left(1-\epsilon+z_1\right)^2\int_0^\infty dx_1dx_2\left(x_1+x_2\right)^{-\epsilon+z_1}\left(x_1+x_2\right)^{-1+\epsilon}\left(1+x_1+x_2\right)^{-1-\epsilon-z_1} \tag{7.28}$$

Now we can introduce another Mellin Barnes representation to factor the sum $(x_1 + x_2)^{-\epsilon+z_1}$ into a product. Then we can immediately integrate over the variables x_1 and x_2 using 3.74. The result consists of many Gamma functions and two Mellin Barnes integrations to be performed:

$$e^{2\epsilon\,\gamma_E}\,x^{-2\epsilon}\int_{-i\infty}^{+i\infty} dz_1 dz_2 \frac{\Gamma(\epsilon)\,\Gamma(2\epsilon)}{\Gamma(1-\epsilon)^2}\frac{\Gamma(\epsilon-z_1)\,\Gamma(-z_1)}{\Gamma(1+\epsilon+z_1)}$$
$$\frac{\Gamma(1-\epsilon+z_1)^2\,\Gamma(\epsilon-z_2)\,\Gamma(-z_2)\,\Gamma(1+z_2)\,\Gamma(1-\epsilon+z_2)\,\Gamma(1+z_1+z_2)}{\Gamma(1+\epsilon+z_2)} \tag{7.29}$$

Now we can carry out the Mellin Barnes integrations. We start with the integration over z_2. We proceed with choosing an integration contour at $-1 < \Re(z_2) < 0$ and pick the residues that appear to the right of this contour. The residues are now due to:

- $\Gamma(-z_2)$
- $\Gamma(\epsilon - z_2)$

which will give us two different contributions. Then after each summation we will be left with the z_1 integration, which we can carry out the same way. For convenience, we will drop the overall factor $e^{2\epsilon\,\gamma_E}\,x^{-2\epsilon}$, which we will recover while expanding our result in ϵ in the end.

First term. Summing the residues due to $\Gamma(-z_2)$ we have:

$$\int_{-i\infty}^{+i\infty} dz_1 \sum_{n_2=0}^{\infty}(-1)^{2-n_2}\frac{\Gamma(\epsilon)\,\Gamma(2\epsilon)}{\Gamma(1-\epsilon)^2}\frac{\Gamma(\epsilon-z_1)\,\Gamma(-z_1)\,\Gamma(1-\epsilon+z_1)^2}{\Gamma(1+\epsilon+z_1)}$$
$$\times\frac{\Gamma(\epsilon-n_2)\,\Gamma(1+n_2)\,\Gamma(1-\epsilon+n_2)\,\Gamma(1+z_1+n_2)}{n_2!\,\Gamma(1+\epsilon+n_2)} \tag{7.30}$$
$$=\frac{\Gamma(\epsilon)\,\Gamma(2\epsilon)}{\Gamma(1-\epsilon)}\int_{-i\infty}^{+i\infty} dz_1$$
$$\times\frac{\Gamma(-1+\epsilon-z_1)\,\Gamma(-z_1)\,\Gamma(1+z_1)\,\Gamma(1-\epsilon+z_1)^2}{\Gamma(1+\epsilon+z_1)} \tag{7.31}$$

Now we can perform the z_1 integration for the expression above. Again we choose the integration contour at $-1 < \Re(z_1) < 0$ then we will have two types of residues to pick due to $\Gamma(-1+\epsilon-z_1)$ and $\Gamma(-z_1)$ if we choose the residues to the right of the contour. The two contributions we get are:

- $\Gamma(-z_1)$:

$$\frac{\Gamma(\epsilon)\,\Gamma(2\epsilon)}{\Gamma(1-\epsilon)}\sum_{n_1=0}^{\infty}(-1)^{2-n_1}\frac{\Gamma(-1+\epsilon-n_1)\,\Gamma(1+n_1)\,\Gamma(1-\epsilon+n_1)^2}{n_1!\,\Gamma(1+\epsilon+n_1)} \tag{7.32}$$

$$= -\,\Gamma(-1+\epsilon)\,\Gamma(-\epsilon)\,\Gamma(2\epsilon)$$
$$\times\,{}_3F_2\Big(\{1,1-\epsilon,1-\epsilon\};\{2-\epsilon,1+\epsilon\};1\Big) \tag{7.33}$$

- $\Gamma(-1+\epsilon-z_1)$:

$$\frac{\Gamma(\epsilon)\,\Gamma(2\epsilon)}{\Gamma(1-\epsilon)}\sum_{n_1=1}^{\infty}(-1)^{2-n_1}\frac{\Gamma(1-\epsilon-n_1)\,\Gamma(n_1)^2\,\Gamma(\epsilon+n_1)}{n_1!\,\Gamma(2\epsilon+n_1)} \tag{7.34}$$

$$= \Gamma(\epsilon)^2\;\psi^{(1)}(2\epsilon) \tag{7.35}$$

These two contributions at hand now we can turn back to Equation 7.29 to pick the residues due to $\Gamma(\epsilon-z_2)$:

Second term. Summing the residues due to $\Gamma(\epsilon-z_2)$ we have:

$$\int_{-i\infty}^{+i\infty} dz_1 \sum_{n_2=0}^{\infty}(-1)^{2-n_2}\frac{\Gamma(\epsilon)\,\Gamma(2\epsilon)}{\Gamma(1-\epsilon)^2}\frac{\Gamma(\epsilon-z_1)\,\Gamma(-z_1)\,\Gamma(1-\epsilon+z_1)^2}{\Gamma(1+\epsilon+z_1)}$$
$$\times\frac{\Gamma(-\epsilon-n_2)\,\Gamma(1+n_2)\,\Gamma(1+\epsilon+n_2)\,\Gamma(1+\epsilon+n_2+z_1)}{n_2!\,\Gamma(1+2\epsilon+n_2)} \tag{7.36}$$

$$= \int_{-i\infty}^{+i\infty} dz_1 \frac{\Gamma(-\epsilon)\,\Gamma(\epsilon)\,\Gamma(1+\epsilon)\,\Gamma(-1+\epsilon-z_1)\,\Gamma(-z_1)\,\Gamma(1-\epsilon+z_1)^2}{\Gamma(1-\epsilon)^2} \tag{7.37}$$

Note that the sum starts from $n_2 = 0$ since $\epsilon \to 0$. Again we now perform the integration over z_1. Choosing the contour same as before and pick the residues to the right we get again two contributions:

- $\Gamma(-z_1)$:

$$\frac{\Gamma(\epsilon)^2}{\epsilon\,\Gamma(-\epsilon)}\sum_{n_1=0}^{\infty}(-1)^{2-n_1}\frac{1}{n_1!}\Gamma(-1+\epsilon-n_1)\,\Gamma(1-\epsilon+n_1)^2 \tag{7.38}$$

$$= -\,\pi\csc(\pi\epsilon)\,(-1+\epsilon)\;\epsilon\,\Gamma(-\epsilon)\,\Gamma(-1+\epsilon)\,\Gamma(\epsilon)^2 \tag{7.39}$$

- $\Gamma(-1+\epsilon-z_1)$:

$$\frac{\Gamma\left(-\epsilon\right)\Gamma\left(\epsilon\right)\Gamma\left(1+\epsilon\right)}{\Gamma\left(1-\epsilon\right)^{2}}\sum_{n_1=1}^{\infty}\left(-1\right)^{2-n_1}\frac{1}{n_1!}\Gamma\left(1-\epsilon-n_1\right)\Gamma\left(n_1\right)^{2} \tag{7.40}$$

$$= -\,\Gamma\left(\epsilon\right)^{2}\psi^{(1)}\left(\epsilon\right) \tag{7.41}$$

Note that the sum starts from $n_1 = 1$. Now we add the four different contributions we obtained and expand in ϵ to arrive at the final result:

$$M^{\mathrm{V}}_{15,2\epsilon} = x^{-2\epsilon}\left(\frac{\pi^2}{12\epsilon^2}+\frac{\zeta_3}{2\epsilon}+\frac{\pi^4}{36}+\left(\frac{31}{36}\pi^2\zeta_3-\frac{19\zeta_5}{2}\right)\epsilon+\mathcal{O}\left(\epsilon^2\right)\right) \tag{7.42}$$

This result is expanded in ϵ therefore do not contain the ϵ dependence fully. However, we calculated some of the integrals with full ϵ dependence, to be consistent we presented the boundary conditions in the next chapters as an expansion in ϵ.

It should be noted that it is important to check these analytical results after taking the residues numerically.

Part III

Automation

Chapter 8

FeynRules

One important aspect of theoretical work in High Energy Physics is to propose New Physics theories. First of all, once a certain level of precision in theoretical computations is reached, the goal is to determine the effects of New Physics that might account for the deviations between theoretical results and experimental results. Secondly, New Physics could also be detected at the colliders directly. Therefore, detailed phenomenological studies on Beyond the Standard Model (BSM) theories are highly crucial. In this respect, FeynRules [46, 48] can play an important role.

FeynRules is a Mathematica package that allows the automatic generation of Feynman rules from the interaction vertices of a given Lagrangian for a particular model. It can be used for various types of BSM models, including the ones that contain superfields [134, 135], and spin two fields [136, 137]. Additionally, FeynRules can also compute the vertices for models with higher dimension operators in their Lagrangian. Once an input file which contains the information about the model is provided, the user can extract the Feynman rules automatically via FeynRules. This is extremely handy, since in extended BSM models, such as in the Minimal Supersymmetric Standard Model (MSSM), doing this by hand is very inconvenient. Recently, FeynRules is extended from producing Feynman rules to analytic computation of two-body decay rates at tree level [49]. In this part of the thesis, we will be particularly interested in the decay module of FeynRules. Before we explain how it works let us briefly describe how to use the package and the main commands of FeynRules.

8.1 The Usage

8.1.1 The Model Input

In order to use the package FeynRules, firstly the user has to provide a file for the model of interest as an input. This model file should be written in Mathematica syntax, within certain specifications that FeynRules provide. It should include the information about the matter fields, gauge fields, parameters of the model, and the type of interactions that are allowed in this model, together with its Lagrangian. With this input FeynRules can generate the Feynman rules for this model via the usage of several commands. To start with, the file can start with a general information about the model, including the name of the model, authors, references to publications, as can be illustrated as:

```
M$ModelName = "my_model";

M$Information = {
   Authors -> {"Ms.D"},
   Institutions -> {"University of Zurich"},
   Emails -> {"x@physik.uzh.ch"},
   Date -> "08.06.2016",
   References -> {"reference 1", "reference 2"},
   URLs -> {"http://feynrules.irmp.ucl.ac.be"},
   Version -> "1.0"
 };
```

When the model is uploaded together with FeynRules package this information will automatically be printed on the Mathematica notebook. After providing this information, the type of interaction indices should be declared:

```
IndexRange[ Index[Colour] ] = Range[3];
IndexRange[ Index[Gluon] ] = NoUnfold[ Range[8] ];
IndexRange[Index[SU2D]] = Unfold[Range[2]];
IndexRange[ Index[SU2W] ] = Unfold[ Range[3] ];
```

Where indices corresponding to two types of interactions are declared:

- `Colour`: fundamental color index ranging from 1 to 3.

- `Gluon`: adjoint color index ranging from 1 to 8.

- `SU2D`: fundamental $SU(2)_L$ index ranging from 1 to 2

- `SU2W`: adjoint $SU(2)_L$ index ranging from 1 to 3.

The `Unfold` command allows the terms in the Lagrangian to be expanded in that certain index. This is necessary in order to obtain the Feynman rules for physical fields if the interactions are written in terms of unphysical fields.

After these declarations the model itself should be implemented which can be done in four different stages: the definition and declaration of the parameters, the definition and declaration of the fields/particles, the gauge groups and the Lagrangian for this model. Now we will review these briefly, but more details can be found here [46, 48].

- The model parameters such as coupling constants, masses, mixing angles should be declared as:

  ```
  M$Parameters = {
      param1 == { options1 },
      param2 == { options2 },
      ...
   };
  ```

  where `param1` and `param2` are the parameters whose properties are declared as replacements with `options`. Let us give an example: consider the strong coupling in QCD. The declaration can be implemented as:

  ```
  gs == {
      TeX              -> Subscript[g,s],
      ParameterType    -> Internal,
      ComplexParameter -> False,
      InteractionOrder -> {QCD, 1},
      Value            -> Sqrt[4 Pi aS],
      ParameterName    -> G,
      Description      -> "Strong coupling constant at the Z pole"
  ```

```
};
```

where the parameter type is specified as internal, meaning that it is not an independent parameter since it depends on another model parameter, α_s. This formula on the other hand is specified with the attribute `Value`. In case of the parameters which are tensors, indices should be declared which can be done with the following command:

```
Indices -> {Index[Scalar], Index[Generation]}
```

which means the parameter carries two types of indices corresponding to `Scalar` and `Generation`. Similar declarations can be done for other parameters of the model, and a complete set of attributes and specifications can be found in Ref. **refs**.

- The fields of the model should be declared in the model file:

```
M$ClassesDescription = {
  F[1] == {
    ClassName -> q,
    ClassMembers-> {d, u, s, c, b, t},
    SelfConjugate -> False,
    Indices -> {Index[Flavor], Index[Colour]},
    FlavorIndex  -> Flavor,
    Mass -> {MQ, {MD, 0}, {MU, 0}, {MS, 0},
                  {MC, 1.25}, {MB, 4.5}, {MT, 174}},
    Width -> {WQ, {WD, 0}, {WU, 0}, {WS, 0},
                  {WC, 0}, {WB, 0}, {WT, 1.6}},
    PDG -> {1,2,3,4,5,6}
  },

  V[1] == {
    ClassName -> G,
    SelfConjugate -> True,
    Indices -> {Index[Gluon]},
    Mass -> 0,
    Width -> 0,
    PDG -> 21
  }}
```

where we defined two particle classes: first one for quarks and the second one is for gluons. The class of quarks include six different quark flavors as indicated in the `ClassMembers` part. We defined two types of indices: `Flavor` and `Colour` for the quarks; where we specified the first one again as the `FlavorIndex`. We also listed the mass `MQ` as a generic tensor parameter with single index, whose entries are mass of the individual quarks. `SelfConjugate` determines whether the particle has an antiparticle or not. In this case setting it to `False` automatically defines an antiparticle by adding a "bar" to the particle's class name. It should be noted that the classes `F` and `V` are specifically reserved for fermions and vector bosons within FeynRules. Also, the Lorentz and spin indices are implemented already in these descriptions for the particle classes `F` and `V` in FeynRules so they do not have to be specified in the model file. Similarly, ghost fields can also be declared the same way, via the usage of the particle class `U`.

Apart from declaring the mass eigenstates, i.e. physical particles, it is also useful to declare the unphysical fields to be able to write the Lagrangian later in a more compact form. To give an example, let us consider the Higgs field in the Standard Model:

```
S[11]  == {
   ClassName -> Phi,
   Unphysical -> True,
   Indices -> {Index[SU2D]},
   FlavorIndex -> SU2D,
   SelfConjugate -> False,
   QuantumNumbers -> {Y -> 1/2},
   Definitions -> {Phi[1] -> -I GP, Phi[2] -> (vev + H + I G0)/Sqrt[2]}
    },
```

this declaration defines a scalar field, which transforms as a doublet in $SU(2)_L$ having the hypercharge, $U(1)_Y$ quantum number 1/2.

- Gauge groups should be declared as:

```
M$GaugeGroups = {
    gaugegroup1 == { options },
    gaugegroup2 == { options },
    ...}
```

With certain attributes, a gauge group can be declared as abelian or non-abelian; and its coupling constants as well as the representations for the gauge group can be implemented:

```
M$GaugeGroups ={ ...,
    U1EM == {Abelian -> True,
             GaugeBoson -> A,
             Charge -> Q,
             CouplingConstant -> ee},
              ...,
    SU3C == {Abelian -> False,
             GaugeBoson -> G,
             StructureConstant -> f,
             CouplingConstant -> gs,
             Representations -> {T, Colour}},
             ...
   }
```

where we declared the gauge groups $U(1)$ and $SU(3)$. As seen from the example, the gauge boson of the group, coupling constants and representations can also be added to the specifications.

- Lastly, the Lagrangian of the model should be written in the model file, in terms of the fields, coupling constants etc. that should have been declared before. Let us give an example. Consider a model with scalars with an $SU(3)$ gauge group, where the scalars are in the adjoint representation:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + D_\mu\phi_i^\dagger D^\mu\phi_i - m^2\phi_i^\dagger\phi_i + \lambda\left(\phi_i^\dagger\phi_i\right)^2 \tag{8.1}$$

$$D_\mu = \partial_\mu - ig_sT^aG^a_\mu \tag{8.2}$$

This Lagrangian can be implemented as the following:

```
LGauge = -1/4 FS[G,mu,nu,a] FS[G,mu,nu,a];

LScalar = DC[phibar[i,a],mu] DC[phi[i,a],mu] - ms^2 phibar[i,a] phi[i,a]
         + lam (phibar[i,a] phi[i,a]) (phibar[j,b] phi[j,b])

L = LGauge + LScalar;
```

where FS stands for the Field Strength, together with DC, the covariant derivative they are hard-coded inside FeynRules so these features can be used directly. The indices are implemented as shown. The parameters, scalar mass m and the self coupling constant λ are written as ms and lam; which should be declared in the parameters section of the model file.

8.1.2 Running FeynRules

In order to run FeynRules the package should be first uploaded to a Mathematica session. For this, the user should indicate the directory that the FeynRules is in and load it as the following:

```
$FeynRulesPath = SetDirectory[...];
<< FeynRules‘
```

where the ellipses indicate the directory of the FeynRules. After loading the FeynRules the model file should be uploaded as well with the command LoadModel:

```
LoadModel[<file.fr>]
```

Note that the extension of the model file is .fr. If the model has an additional file, it could also be uploaded after the main file. Now we can describe how to use FeynRules to extract Feynman rules. The Feynman rules can be obtained by using the command FeynmanRules with the Lagrangian of the model being the argument:

```
verts = FeynmanRules[L];
```

Upon running this command the vertices of the Lagrangian will be derived by FeynRules, which will then be printed on the screen. Note that by assigning a name verts, we have the Mathematica store the vertices internally. An important command is SelectVertices, which allows one to extract a particular vertex that one needs:

```
vertsGluon = SelectVertices[ verts, SelectParticles->{{G,G,G},{G,G,G,G}}];
```

With this, we tell FeynRules to explicitly give the three-gluon and four-gluon vertices from the already computed vertices, verts. FeynRules has various more options which can be found here [48].

8.2 The Decay Module of FeynRules

One important way to analyse New Physics models is via Monte Carlo simulations, which allows one to generate events and numerically compute the cross sections. There are various stages to this simulation process, involving hard processes, parton showers, hadronisation etc. The hard processes, i.e. the partonic cross section, require the computation of matrix elements and integration over the phase space. This requires firstly the implementation of the Feynman rules of the model of interest into the matrix element generators. At leading order, the cross section for a particular process can be computed by the matrix element generator and at next to leading order analytic results for matrix elements can be implemented into the Monte Carlo generator with the divergences already subtracted via a suitable subtraction method such that a stable numerical result can be obtained. In order to obtain a full description of a particular event, one has to take into account not only production, but also the subsequent decay of the corresponding particles. This requires the total width of the particles to be known. In principle, the computation of tree level decay rates can be handled by the matrix element generator. However, the numerical result should then be given back as an input in the model file to the matrix element generator. This is highly inefficient since the total decay width depends on various parameters of the model and it then has to be reevaluated for every benchmark scenario; and the resulting number should be plugged back into the model information as a parameter. In addition to this, the implementation of the Feynman rules of a model into a matrix element generator is often quite tedious to do by hand, and requires automation.

FeynRules has an interface to various matrix element generators, such as MadGraph [138–142], and many others [143–147], which allows one to export Feynman rules of a given model that are generated via FeynRules as a Universal FeynRules Output format (UFO) [47]. In addition to this, with the new FeynRules decay module the decay width calculation can also be automated. In particular, the tree level, two body decay widths for a generic New Physics model can be computed analytically now with FeynRules. The analytical expressions for decay widths are then included to the UFO output, which can then be dynamically used in a matrix element generator for different benchmark scenarios with different model parameters. Two-body decays may not be sufficient for an accurate estimation of a total width of a particle, computing the full width analytically is often out of reach. For this purpose, an additional module for MadGraph5_aMC@NLO [141–143, 148–150] which is called MADWIDTH [49] is created. It determines automatically the required final state multiplicity to reach a given precision on the total width, and it is possible with MADWIDTH to compute three-body decay widths at tree level. In what follows we will explain how the decay module of FEYNRULES works. Later, we will briefly present MADWIDTH. We will give some examples for various decay widths that we computed via FEYNRULES and MADWIDTH, following ref. [49].

Let us first give some formulae necessary to compute the two body decay width at tree level. At leading order two-body partial decay width of a particle is given by:

$$\Gamma = \frac{1}{2|M|N} \int d\Phi_2 |\mathcal{M}|^2 \tag{8.3}$$

where N is the symmetry factor, M is the mass of the decaying particle, and $d\Phi_2$ is the two particle phase space measure. $|\mathcal{M}|^2$ is the averaged squared matrix element, which is basically the three-point vertex, that can be written as:

$$|\mathcal{M}|^2 = \mathcal{V}^{a_1 a_2 a_3}_{l_1 l_2 l_3} \mathcal{P}_1^{l_1 l_1'} \mathcal{P}_2^{l_2 l_2'} \mathcal{P}_3^{l_3 l_3'} \left(\mathcal{V}^*\right)^{a_1 a_2 a_3}_{l_1' l_2' l_3'} \tag{8.4}$$

where the color and spin indices of the particle i are denoted by $l_i^{(')}$ and a_i. In addition, $\mathcal{P}_i$ denotes the polarization tensor of the particle i, which depends on its spin and its mass. $\mathcal{V}$ is the three-point vertex. For two body decays, the matrix element squared is independent of the phase space measure parameters, therefore the computation of the decay width is straightforward and given by:

$$\Gamma = \frac{\sqrt{\lambda\left(M^2, m_1^2, m_2^2\right)}\, |\mathcal{M}|^2}{16\pi N |M|^3} \tag{8.5}$$

where $\lambda\left(M^2, m_1^2, m_2^2\right) = \left(M^2 - m_1^2 - m_2^2\right)^2 - 4m_1^2 m_2^2$; and m_1 and m_2 are masses of the particles that the heavy particle decays into. The reason why FEYNRULES does not yet compute three body decay width at tree level is because the matrix element does not decouple from the phase space and needs to be integrated over it. This may lead to divergences for massless particles in the final state. Therefore the implementation is not trivial.

Using the Feynman rules that FEYNRULES can already determine, the analytic computation of the two body decay width at tree level for new physics models is then easy in FEYNRULES. The computation of the width is done by the command

`ComputeWidths`:

```
verts = FeynmanRules[L];
decays = ComputeWidths[verts];
```

where `L` is variable that contains the Lagrangian for the model, and `verts` is the variable that the Feynman rules that are extracted via FEYNRULES is stored into. The command `ComputeWidths` will give a list of decay widths as an output:

`{{`ϕ_1, ϕ_2, ϕ_3`}, ` $\Gamma_{\phi_1 \to \phi_2 \phi_3}$`}`

This command selects the three-point vertices from the vertices that are already computed, with at least one massive particle, irrespective of whether the channel is kinematically allowed or not. The reason is simple: for a given new physics model, the input parameters of the model can change, and the decay channel might be open in one benchmark scenario, but not open in another. Additionally, the selected vertices should not contain any ghost fields or Goldstone bosons since the decay processes would not involve these fields. The output will be given in analytic form. One can also extract the information about specific decay channels using the following commands:

`PartialWidth[{`ϕ_1, ϕ_2, ϕ_3`}, decays];`
`TotWidth[`ϕ_1`, decays];`
`BranchingRatio[{`ϕ_1, ϕ_2, ϕ_3`}, decays];`

The command `PartialWidth` first checks the numerical values of the masses of the particles involved in the process, and returns the decay width for $\phi_1 \to \phi_2\phi_3$ if it is allowed. It should be noted that all the partial decay widths that are computed via `ComputeWidths` are stored into the global variable `FR$PartialWidth`. `TotWidth` and `BranchingRatio` return the analytical result for the total width and branching ratios. Even though these outputs are analytical expressions, the numerical values can easily be obtained with the usual MATHEMATICA command `NumericalValue[]`. Finally, it is important to emphasise that the numerical decay width information of the particles can be inserted or updated in the model information via the following command:

```
UpdateWidths[ decays ];
```

which is crucial for the matrix element generators. As we already mentioned, FEYN-RULES has an interface to many matrix element generators, i.e. a file involving the model information in UFO format can be extracted from FEYNRULES to be used for example in MadGraph. The UFO format is also extended to include the analytic expression for decay widths, since a numerical value is only valid for particular choice of parameters of the model. Issuing the command:

```
WriteUFO[];
```

generates various files in UFO format that contain the model information, including also the analytical expressions for decay widths in a file called `decays.py`. For example, the decay information for the Higgs boson can be included in this file as:

```
Decay_H = Decay(name = 'Decay_H',
        particle = P.H,
        partial_widths = {(P.W__minus, P.W__plus): 'Γ_{H→W+W-}',
                          (P.Z, P.Z.): 'Γ_{H→ZZ}',
                          (P.b, P.b__tilde__): 'Γ_{H→bb̄}',
                          (P.t, P.t__tilde__): 'Γ_{H→tt̄}'
   )
```

where $\Gamma_{H\to ij}$ stands for the analytic expressions for the corresponding partial decay widths.

Let us give a concrete example, and consider the BSM model, Strongly Interacting Light Higgs (SILH) [151]. This model was established before the discovery of the Higgs boson; however since it involves higher dimension operators, it is a good validation for the decay module. In this model, there is a new sector which is responsible for the Electroweak Symmetry Breaking (EWSB). The interactions of this sector is characterised by the mass scale m_ρ and the gauge coupling g_ρ. The strength of this new coupling is assumed to be larger than the SM couplings, and therefore the corresponding interaction is referred to as *strong*. Higgs doublet belongs to the strong sector, and Higgs boson emerges as a Goldstone boson of the spontaneously broken symmetry of the strong sector. With these assumptions then, an effective Lagrangian involving the interactions of the SM fields and the Higgs field is constructed. More details of this model is beyond the scope of this thesis, however below we will give an example of how FeynRules is used to obtain the Higgs-gluon-gluon vertex from this model, and the decay width of $H \to gg$. In this model, the piece of the Lagrangian that describe the interaction of the Higgs and gluon fields is a dimension-6 operator and given by:

$$\mathcal{L} = \frac{c_g\, g_s^2\, y_t^2\, \xi}{16\pi^2\, v^2 g_\rho^2}\, H^\dagger H\, F^a_{\mu\nu} F^{a\mu\nu} \tag{8.6}$$

where H is the Higgs doublet, $F^a_{\mu\nu}$ is the field strength tensor for the gluon field; y_t is the Yukawa coupling for top quark, g_s is the strong coupling of QCD, v is the vacuum expectation value of the Higgs field. c_g and g_ρ are the new couplings resulting from the strong sector, and ξ is the parameter characterising the strong sector. It is particularly the ratio of the Higgs vacuum expectation value and the scale of the strong sector f: $\xi = v^2/f^2$.

From this Lagrangian, the Hgg vertex extracted via FeynRules is:

$$\mathcal{V}^{\mu_1\mu_2}_{a_1 a_2} = \frac{i c_g\, g_s^2\, \xi\, y_t^2\, \delta_{a_1 a_2}\, (p_1^{\mu_2} p_2^{\mu_1} - g^{\mu_1\mu_2}\, p_1 \cdot p_2)}{4 g_\rho^2\, \pi^2\, v} \tag{8.7}$$

where p_1 and p_2 are the momenta of the gluons; a_1, a_2 and μ_1, μ_2 are color and Lorentz indices of the gluons respectively. The decay width of $H \to gg$ is easily computed via FeynRules, which gives:

$$\Gamma_{H\to gg} = \frac{c_g^2\, g_s^4\, m_H^3\, \xi^2\, y_t^4}{128\, g_\rho^4\, \pi^5\, v^2} \tag{8.8}$$

The numerical result of this expression is given in Table 8.2. The rest of the new terms in the SILH Lagrangian also contain dimension-6 operators, describing new tree-level interactions of such as $H\gamma\gamma$ and $H\gamma Z$. The numerical values for the corresponding decay widths are also given in Table 8.2. We also compute all other possible two body, tree level decays which we do not display here. These checks validate that the automatic computation of FeynRules can be used in a LO Monte Carlo generator.

The decay module of FeynRules can automatically compute the two-body decay widths; however there are cases where two body or tree level decay widths are not sufficient to make a reliable prediction. For this reason, the MadGraph5_aMC@-NLO module MadWidth [49] is created. MadWidth can compute tree level, N body decay widths. As in the two-body decay widths, these can also be computed via matrix element generators, i.e. directly with MadGraph5_aMC@NLO. However these methods have limits, such that cascade decays and real radiations are also included in the decay widths. The algorithm of MadWidth module on the other hand can distinguish between these processes, and also make a valid estimation of which multiplicities of the final state particles are relevant to the total decay width.

In order to validate the decay module of FeynRules, we performed various computations [49]. We worked in the frameworks of Standard Model (SM) and two different BSM models for this: Strongly Interacting Light Higgs (SILH) [151], and the SPS1a MSSM benchmark scenario [152]. We performed all possible tree level, two body decay rates in these models and checked our results to the numerically computed results via MadWidth module of MadGraph5_aMC@NLO. A selection of numerical results can be found in Table 8.1, Table 8.2 and Table 8.3 for the Standard Model, the SILH model and the MSSM, respectively.

Table 8.1: Selection of partial decay widths in the framework of the Standard Model, as computed by FeynRules and MadWidth.

Decay Mode	FeynRules [GeV]	MadWidth [GeV]
$H \to b\bar{b}$	0.005390	0.005391
$H \to \tau\bar{\tau}$	0.0002587	0.0002587
$H \to c\bar{c}$	0.0003967	0.0003967
$W^+ \to e^+\nu_e$	0.2225	0.2225
$W^+ \to \tau^+\nu_\tau$	0.2223	0.2224
$W^+ \to u\bar{d}$	0.6336	0.6336
$W^+ \to c\bar{s}$	0.6333	0.6334
$W^+ \to c\bar{d}$	0.03401	0.03402
$W^+ \to u\bar{s}$	0.03403	0.03403
$Z \to e^-e^+$	0.08329	0.08329
$Z \to \tau^-\tau^+$	0.0831	0.0831
$Z \to \nu_e\bar{\nu}_e$	0.1658	0.1659
$Z \to u\bar{u}$	0.2841	0.2842
$Z \to d\bar{d}$	0.3667	0.3667
$Z \to c\bar{c}$	0.2838	0.2839
$Z \to b\bar{b}$	0.3627	0.3628
$t \to bW^+$	1.466	1.467

Table 8.2: Higgs boson partial decay widths in the framework of the SILH model, as computed by FeynRules and MadWidth.

Decay Mode	FeynRules [GeV]	MadWidth [GeV]
$H \to \gamma\gamma$	6.447e-10	6.447e-10
$H \to gg$	7.523e-06	7.524e-06
$H \to \gamma Z$	4.026e-11	4.026e-11

Table 8.3: Selection of partial decay widths in the framework of the SPS1a MSSM scenario, as computed by FeynRules and MadWidth.

Decay Mode	FeynRules [GeV]	MadWidth [GeV]
$\tilde{\chi}_4^0 \to \tilde{\chi}_1^+ W^-$	0.6451	0.6451
$\tilde{\chi}_4^0 \to \tilde{\chi}_1^0 Z$	0.05567	0.05568
$\tilde{\chi}_2^+ \to \tilde{\chi}_1^0 W^+$	0.1682	0.1683
$\tilde{\chi}_2^+ \to \tilde{\chi}_1^+ Z$	0.5755	0.5756
$\tilde{u}_6 \to \tilde{u}_3 Z$	1.39	1.4

Part IV

Outlook

Chapter 9

Conclusion

In this work, we described how we calculated the top-bottom interference contribution to the inclusive cross section of the Higgs boson production in gluon fusion at NLO in QCD as an expansion in the bottom quark mass. The cross section has two pieces to it: real-virtual and double-virtual contributions. We gave our results for the cross section, as well as matrix elements in terms of master integrals that appear in real-virtual and double-virtual pieces. We also described how we determined boundary conditions by computing different regions of the master integrals, with the methods we use being differential equations, and the method of Mellin Barnes representation. Our results for the boundary conditions are presented in Appendix C.

With the latest developments in the precision Higgs physics, the bottom quark mass effects became crucial in reducing the theoretical uncertainties. Top-bottom interference is interesting because it contains the first order contribution in the bottom quark mass to the cross section for gluon fusion at a given α_s order. Furthermore, our results are presented as an expansion up to order $\mathcal{O}(\epsilon)$ such that they can also be used for the renormalization at NNLO correction. In addition to this, the method that we use here in order to obtain an expansion in the bottom quark mass will also be useful to perform the computation for the three loop correction (NNLO), since the three loop computation with full bottom quark mass dependence is not feasible at the moment. By determining this NNLO contribution the theoretical uncertainty is expected to be reduced by 0.6% [33].

In high energy physics, precision computations constitute an important part of the theoretical improvements. With precision physics, the aim is to make better predictions for observables and to reduce the theoretical uncertainties. Another important piece of high energy physics is to propose new physics models, i.e. beyond the Standard Model (BSM) theories, that could account for the deviations between theoretical predictions and experimentally measured results; or the predicted particles of which could be directly detected at the colliders. Therefore, developing new physics models and analysing their phenomenological implications is very important. In the last chapter, we introduced the MATHEMATICA package FEYNRULES that provides a first step to this kind of analysis: it allows the automatic computation of interaction vertices for a model of choice which should be given as an input by the user. Obtaining Feynman rules for a generic model can be very useful if one wants to perform calculations and phenomenological studies on a New Physics model one is interested in. In addition to this, FEYNRULES has a decay module now such that it is also possible to automatically compute a tree-level two body decay rate via FEYNRULES. The automatic computation of decay rates for new physics models is crucial, for the decay rate informations need to be included when one would like to use Monte Carlo generators to simulate new physics theories. FEYNRULES has an interface, called Universal Feynrules Output (UFO), to many matrix element generators. One can export the model information, together with the decay rates in UFO format, to use in Monte Carlo programs.

Part V

Appendix

Appendix A

Matrix Elements

A.1 Matrix Elements for Double-Virtual Contribution

$$
\begin{aligned}
A_{\mathrm{V}} = \Bigg\{ & -\frac{\epsilon-1}{2x^2(x+1)^2(1-4\epsilon)^2(1-2\epsilon)^2\epsilon(3\epsilon-1)}\Bigg(C_A\Big[(x \\
& + 1)^2 \left(128(x-1)^2\left(10x^2-17x+10\right)\epsilon^8 - 32(x-1)^2\left(88x^2-173x+88\right)\epsilon^7\right. \\
& + 16\left(107x^4-510x^3+1094x^2-510x+107\right)\epsilon^6 \\
& + 8\left(25x^4-255x^3-932x^2-255x+25\right)\epsilon^5 \\
& + \left(-538x^4+6992x^3+1428x^2+6992x-538\right)\epsilon^4 \\
& + \left(185x^4-4720x^3-2034x^2-4720x+185\right)\epsilon^3 \\
& - 8\left(3x^4-204x^3-206x^2-204x+3\right)\epsilon^2 + \left(x^4-304x^3-482x^2-304x+1\right)\epsilon \\
& \left. + 24x(x+1)^2\right)\Big] + C_F\Big[\\
& - 4\epsilon\left(4\epsilon^2-5\epsilon+1\right)\Big(x^6-12x^5-25x^4+8x^3-5\left(x^2-1\right)^2\left(5x^2-52x+5\right)\epsilon^3 \\
& + 16\left(x^2-1\right)^2\left(10x^2-11x+10\right)\epsilon^5 - 4\left(x^2-1\right)^2\left(33x^2-17x+33\right)\epsilon^4 - 25x^2 \\
& - 4\left(3x^6-25x^5-37x^4+38x^3-37x^2-25x+3\right)\epsilon \\
& + \left(44x^6-282x^5-236x^4+564x^3-236x^2-282x+44\right)\epsilon^2 - 12x + 1\Big)\Big]\Bigg)\Bigg\} M_1^{\mathrm{V}}
\end{aligned}
$$

$$+\Bigg\{-\frac{2}{(x-1)^2(x+1)^2(1-4\epsilon)^2(\epsilon-1)(2\epsilon-1)}\Bigg(C_A\Big[(x$$
$$+1)^2\left(-16\left(13x^2-50x+13\right)\epsilon^5+10\left(17x^2-114x+17\right)\epsilon^4\right.$$
$$+\left(66x^2+796x+66\right)\epsilon^3-\left(127x^2+362x+127\right)\epsilon^2$$
$$\left.+\left(49x^2+102x+49\right)\epsilon+32(x-1)^2\epsilon^7+24(x-1)^2\epsilon^6-6(x+1)^2\right)\Big]$$
$$+C_F\Big[-2\left(4\epsilon^2-5\epsilon+1\right)\Big(24\left(x^2-1\right)^2\epsilon^5-30\left(x^2-1\right)^2\epsilon^4+37\left(x^2-1\right)^2\epsilon^3$$
$$+(x-1)^2\left(x^2-6x+1\right)-4\left(7x^4+12x^3-22x^2+12x+7\right)\epsilon^2$$
$$+2\left(x^4+24x^3-34x^2+24x+1\right)\epsilon\Big)\Big]\Bigg)\Bigg\}M_3^{\mathrm{V}}$$

$$+\left\{C_A\left[-\frac{8x\left(\epsilon^3+2\epsilon^2-3\epsilon+1\right)}{(x-1)^2(\epsilon-1)}\right]\right\}M_4^{\mathrm{V}}$$

$$-\Bigg\{\Bigg(C_A\Big[-64(x-1)^2\left(5x^2-14x+5\right)\epsilon^8-2x\left(23x^2+42x+23\right)\epsilon$$
$$+16\left(39x^4-192x^3+298x^2-192x+39\right)\epsilon^7$$
$$-16\left(17x^4-106x^3+188x^2-106x+17\right)\epsilon^6$$
$$-2\left(59x^4-112x^3+786x^2-112x+59\right)\epsilon^5$$
$$+\left(105x^4-224x^3+2894x^2-224x+105\right)\epsilon^4$$
$$-2\left(10x^4+131x^3+678x^2+131x+10\right)\epsilon^3$$
$$+\left(x^4+192x^3+398x^2+192x+1\right)\epsilon^2+4x(x+1)^2\Big]$$
$$+C_F\Big[4\epsilon\left(4\epsilon^2-5\epsilon+1\right)\Big(8(x-1)^2\left(5x^2-14x+5\right)\epsilon^5$$
$$+\left(-23x^4+136x^3-194x^2+136x-23\right)\epsilon^4$$
$$-4\left(3x^4-3x^3+14x^2-3x+3\right)\epsilon^3+\left(8x^4-8x^3+68x^2-8x+8\right)\epsilon^2$$
$$-\left(x^4+8x^3+22x^2+8x+1\right)\epsilon+2x(x+1)^2\Big)\Big]\Bigg)\Bigg\}M_5^{\mathrm{V}}$$

$$
\begin{aligned}
&+\Bigg\{-\frac{2(x+1)^2\epsilon}{(x-1)^4(1-2\epsilon)^2(\epsilon-1)(3\epsilon-1)(4\epsilon-1)}\Bigg(C_A\Big[\\
&-16\left(7x^2-15x+7\right)\epsilon^4+4\left(8x^2-33x+8\right)\epsilon^3+6\left(5x^2-18x+5\right)\epsilon^2\\
&-5\left(3x^2-26x+3\right)\epsilon+x^2+64(x-1)^2\epsilon^5-26x+1\Big]\\
&+C_F\Big[-4\left(4\epsilon^2-5\epsilon+1\right)\left(\left(-3x^2+10x-3\right)\epsilon^2\right.\\
&\left.+\left(-3x^2+x-3\right)\epsilon+8(x-1)^2\epsilon^3+(x-1)^2\right)\Big]\Bigg)\Bigg\}M_6^{\mathrm{V}}
\end{aligned}
$$

$$
\begin{aligned}
&-\Bigg\{\frac{2}{(x-1)^4(1-4\epsilon)^2(\epsilon-1)\epsilon}\Bigg(C_A\Big[32(x-1)^2\left(x^2-4x+1\right)\epsilon^7\\
&-8(x-1)^2\left(5x^2-28x+5\right)\epsilon^6-6(x+1)^2\left(x^2-4x+1\right)\\
&+16\left(x^4-8x^3-34x^2-8x+1\right)\epsilon^5\\
&-2\left(61x^4-160x^3-602x^2-160x+61\right)\epsilon^4\\
&+2\left(119x^4-310x^3-546x^2-310x+119\right)\epsilon^3\\
&+\left(-171x^4+422x^3+730x^2+422x-171\right)\epsilon^2\\
&+\left(53x^4-118x^3-270x^2-118x+53\right)\epsilon\Big]\\
&+C_F\Big[-2\epsilon\left(4\epsilon^2-5\epsilon+1\right)\left(8(x-1)^2\left(x^2-4x+1\right)\epsilon^4\right.\\
&+2(x-1)^2\left(3x^2+4x+3\right)\epsilon^3\\
&-(x-1)^2\left(7x^2-12x+7\right)\epsilon^2+2(x+1)^2\left(x^2-4x+1\right)\\
&\left.+\left(-3x^4+4x^3+30x^2+4x-3\right)\epsilon\right)\Big]\Bigg)\Bigg\}M_7^{\mathrm{V}}\\
&+\Bigg\{\frac{4x^2}{(x-1)^6(1-4\epsilon)^2\epsilon^2(2\epsilon-1)}\Bigg(C_A\Big[(2\epsilon\\
&-1)\left(4\left(3x^2+2x+3\right)\epsilon^3-\left(47x^2+58x+47\right)\epsilon^2\right.\\
&\left.+\left(26x^2+44x+26\right)\epsilon+24(x-1)^2\epsilon^5-6(x-1)^2\epsilon^4-4(x+1)^2\right)\Big]\\
&+C_F\Big[-2\epsilon(4\epsilon-1)\left(-\left(x^2+6x+1\right)\epsilon+12(x-1)^2\epsilon^4\right.\\
&\left.-5(x-1)^2\epsilon^3-4(x-1)^2\epsilon^2+(x+1)^2\right)\Big]\Bigg)\Bigg\}M_8^{\mathrm{V}}
\end{aligned}
$$

$$+\Bigg\{C_A\Big[-\frac{4x\left(\epsilon^3+2\epsilon^2-3\epsilon+1\right)\left((x-1)^2\epsilon-(x+1)^2\right)}{(x-1)^4(\epsilon-1)\epsilon}\Big]\Bigg\}M_9^{\mathrm{V}}$$

$$+\Bigg\{-\frac{2(\epsilon-1)}{(x-1)^2\left(8\epsilon^2-6\epsilon+1\right)}\Bigg(C_A\Big[(2\epsilon-1)\left(-x^2\right.$$

$$\left.+16(x-1)^2\epsilon^3-4(x-1)^2\epsilon^2-4(x-1)^2\epsilon-6x-1\right)\Big]$$

$$+C_F\Big[-2(4\epsilon-1)\left(-8\left(x^2-x+1\right)\epsilon^2+2\left(x^2+8x+1\right)\epsilon\right.$$

$$\left.-x^2+10(x-1)^2\epsilon^3-14x-1\right)\Big]\Bigg)\Bigg\}M_{10}^{\mathrm{V}}$$

$$-\Bigg\{\frac{\epsilon-1}{(x-1)^2\left(8\epsilon^2-6\epsilon+1\right)}\Bigg(C_A\Big[-2\left(7x^2-38x+7\right)\epsilon+x^2$$

$$-64(x-1)^2\epsilon^4+48(x-1)^2\epsilon^3+16(x-1)^2\epsilon^2-18x+1\Big]$$

$$+C_F\Big[4(4\epsilon-1)\left(x^2+8(x-1)^2\epsilon^3\right.$$

$$\left.-3(x-1)^2\epsilon^2-3(x-1)^2\epsilon-6x+1\right)\Big]\Bigg)\Bigg\}M_{11}^{\mathrm{V}}$$

$$-\Bigg\{-\frac{4(3\epsilon-2)}{(x-1)^2(1-4\epsilon)^2(\epsilon-1)\epsilon}C_A\Big[2\left(x^2+94x+1\right)\epsilon^4$$

$$-4\left(15x^2+46x+15\right)\epsilon^3+\left(89x^2+134x+89\right)\epsilon^2$$

$$-\left(41x^2+70x+41\right)\epsilon+16(x-1)^2\epsilon^6-12(x-1)^2\epsilon^5+6(x+1)^2\Big]$$

$$+C_F\Big[-2\epsilon\left(4\epsilon^2-5\epsilon+1\right)\left(4(x-1)^2\epsilon^3\right.$$

$$\left.+5(x-1)^2\epsilon^2-(x-1)^2\epsilon-2(x+1)^2\right)\Big]\Bigg\}M_{12}^{\mathrm{V}}$$

$$+\left\{ C_F \left[\frac{8x\left(x^2+(x+1)^2\epsilon^2-2(x+1)^2\epsilon+1\right)}{(x^2-1)^2} \right] \right\} M_{13}^{\mathrm{V}}$$

$$-\left\{ \frac{4x}{(x-1)^6\left(8\epsilon^2-6\epsilon+1\right)} \left(C_A \left[(x+1)^2\epsilon(2\epsilon \right.\right.\right.$$

$$\left. -1)\left(4(x-1)^2\epsilon^3-(x-1)^2\epsilon^2-(x+1)^2\right) \right]$$

$$-C_F\left[(4\epsilon-1)\left(4\left(x^2-1\right)^2\epsilon^4-3\left(x^2-1\right)^2\epsilon^3\right.\right.$$

$$\left.\left.\left.\left.+(x+1)^2\left(x^2+1\right)-2\left(x^4+x^3+4x^2+x+1\right)\epsilon\right)\right]\right)\right\} M_{14}^{\mathrm{V}}$$

$$+\left\{ \left(C_A-2C_F\right)\left[\frac{4x\epsilon\left(2\epsilon^2-\epsilon-1\right)}{(x-1)^2}\right] \right\} M_{15}^{\mathrm{V}}$$

$$-\left\{ \frac{4x}{(x-1)^4(4\epsilon-1)} \left(C_A \left[(2\epsilon \right.\right.\right.$$

$$\left. -1)\left(4(x-1)^2\epsilon^3-(x-1)^2\epsilon^2-(x+1)^2\right) \right]$$

$$\left.\left.-C_F\left[\left(4\epsilon^2-5\epsilon+1\right)\left(4(x-1)^2\epsilon^2+2(x-1)^2\epsilon-(x+1)^2\right)\right]\right)\right\} M_{16}^{\mathrm{V}}$$

$$+\left\{ C_A\left[\frac{x\left(-\left(7x^2+10x+7\right)\epsilon+4(x-1)^2\epsilon^2+2(x+1)^2\right)}{(x-1)^4(4\epsilon-1)}\right] \right\} M_{17}^{\mathrm{V}}$$

A.2 Matrix Elements for Real-Virtual Contribution

$$
\begin{aligned}
A_{\mathrm{R}} = \Bigg\{ & -\frac{1}{(z-1)z(z+1)\epsilon(2\epsilon-1)(4\epsilon-1)\left(1-4r\right)^2\left(4r-z+1\right)} \\
& \Bigg[4\left(z(z+1)\left(1-4r\right)^2\left(\left(5z^2+2z+1\right)r+8(z-1)r^2+z\left(-2z^2+3z-3\right)\right)\right. \\
& + z\epsilon\left(-384\left(z^2-1\right)r^4-16\left(39z^3+9z^2-7z+71\right)r^3\right. \\
& + 8\left(29z^4+31z^3-3z^2+65z+54\right)r^2+\left(-124z^4+33z^3+7z^2-261z-23\right)r \\
& \left. + 15z^4-8z^3-4z^2+34z-5\right) \\
& + 16\epsilon^4\left(128(z-1)^2\left(z^2+3z+2\right)r^3+2\left(12z^5-27z^3+10z^2+21z-8\right)r\right. \\
& \left. - 4\left(11z^5+16z^4-19z^3-26z^2+10z+16\right)r^2-3z^5+z^4+8z^3-6z^2-6z+4\right) \\
& - 2\epsilon^3\left(128\left(5z^4+9z^3-27z^2+z+20\right)r^3\right. \\
& - 8\left(43z^5+68z^4-109z^3-46z^2+116z+80\right)r^2 \\
& + 4\left(52z^5-2z^4-115z^3+106z^2+103z-40\right)r-25z^5+4z^4+70z^3-86z^2 \\
& \left. - 47z+40\right)+\epsilon^2\left(256z\left(z^2-1\right)r^4+32\left(41z^4+11z^3-77z^2+105z+32\right)r^3\right. \\
& - 8\left(71z^5+84z^4-123z^3+124z^2+204z+32\right)r^2 \\
& + 2\left(166z^5-71z^4-131z^3+429z^2+87z-32\right)r-39z^5+23z^4+40z^3 \\
& \left.\left. - 131z^2+7z+16\right)\right)\Bigg]\Bigg\} M_1^{\mathrm{R}}
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg\{\\
&-\frac{1}{\epsilon(2\epsilon-1)(4\epsilon-1)\left(4r-z+1\right)\left(z-1\right)(z+1)\left(z-4rz\right)^2}\Bigg[512(1-2\epsilon)^2z^2\left(z^2-1\right)r^5\\
&-64\left(51z^5-43z^4+21z^3+51z^2-32z+256\epsilon^4(z-1)^3\left(2z^2+3z+1\right)\right.\\
&-8\epsilon\left(39z^5-34z^4+23z^3+40z^2-20z+24\right)-16\epsilon^3\left(45z^5-61z^4-13z^3+73z^2+16z-12\right)\\
&\left.+4\epsilon^2\left(161z^5-165z^4+79z^3+189z^2-16z+56\right)+32\right)r^4\\
&+32\left(34z^6-11z^5+10z^4+55z^3+16z+16\epsilon^4\left(24z^6-9z^5-33z^4+17z^3+33z^2-8\right)\right.\\
&-4\epsilon^3\left(158z^6-49z^5-57z^4+129z^3+187z^2+96z-24\right)\\
&-\epsilon\left(230z^6-53z^5+97z^4+345z^3+45z^2+144z+96\right)\\
&\left.+2\epsilon^2\left(269z^6-62z^5+92z^4+330z^3+163z^2+208z+56\right)+16\right)r^3-4\left(16z^7+96z^6-141z^5\right.\\
&+153z^4+117z^3-97z^2+128z+32\epsilon^4\left(2z^7+49z^6-63z^5-15z^4+101z^3-26z^2-8z+8\right)\\
&-8\epsilon^3\left(28z^7+276z^6-251z^5+31z^4+447z^3+29z^2+8z+24\right)\\
&-4\epsilon\left(29z^7+159z^6-173z^5+232z^4+158z^3-85z^2+200z-48\right)\\
&\left.+2\epsilon^2\left(130z^7+787z^6-636z^5+750z^4+866z^3+23z^2+592z-112\right)-32\right)r^2\\
&+2\left(8\left(8z^7+51z^6-76z^5-18z^4+124z^3-25z^2-32z+16\right)\epsilon^4\right.\\
&-2\left(106z^7+197z^6-152z^5-142z^4+422z^3+305z^2-288z+48\right)\epsilon^3\\
&+\left(245z^7+57z^6+122z^5+26z^4+33z^3+717z^2-192z-112\right)\epsilon^2\\
&-\left(113z^7-35z^6+44z^5+136z^4-81z^3+199z^2+48z-96\right)\epsilon\\
&\left.+2\left(8z^7-3z^6-3z^5+16z^4-3z^3+5z^2+8z-8\right)\right)r\\
&-z\left(z^2-1\right)\left(4\left(5z^4-5z^3+19z^2-27z+16\right)\epsilon^4\right.\\
&+\left(-61z^4+101z^3-267z^2+315z-176\right)\epsilon^3+2\left(33z^4-70z^3+153z^2-158z+84\right)\epsilon^2\\
&\left.+\left(-29z^4+69z^3-133z^2+125z-64\right)\epsilon+2\left(2z^4-5z^3+9z^2-8z+4\right)\right)\Bigg]\Bigg\}M_2^{\mathrm{R}}
\end{aligned}
$$

$$
+\Bigg\{-\frac{1}{(8\epsilon^2-6\epsilon+1)\,(1-4r)^2}\Bigg[2(z-1)\epsilon r\,\Big(4\epsilon^2\,(8r-z-1)
$$

$$
+\epsilon\,\Big(64(z-7)r^2-8(z-20)r+5z-19\Big)-48(z-3)r^2+2(9z-41)r
$$

$$
+128r^3-3z+11\Big)\Bigg]\Bigg\}\,M_3^{\mathrm{R}}+\Bigg\{\frac{1}{(z-1)^2z(z+1)(\epsilon-1)\epsilon(2\epsilon-3)(4\epsilon-1)\,(1-4r)^2}
$$

$$
\Bigg[2r\,\Big(16z\epsilon^5\,\Big(16\,\Big(9z^4+22z^3-20z^2-6z+27\Big)\,r^2-8\,\Big(9z^4+20z^3-16z^2-8z+27\Big)\,r
$$

$$
+7z^4+24z^3-18z^2-8z+27\Big)-4\epsilon^4\,\Big(16\,\Big(93z^5+266z^4-216z^3-42z^2+219z+64\Big)\,r^2
$$

$$
-8\,\Big(97z^5+224z^4-144z^3-76z^2+219z+64\Big)\,r+67z^5+284z^4-174z^3
$$

$$
-76z^2+219z+64\Big)+4\epsilon^3\,\Big(16\,\Big(91z^5+241z^4-177z^3+43z^2+18z+176\Big)\,r^2
$$

$$
-8\,\Big(102z^5+172z^4-69z^3-10z^2+21z+176\Big)\,r+128(z-1)^2z(z+1)r^3+63z^5+251z^4
$$

$$
-109z^3-11z^2+22z+176\Big)-2\epsilon^2\,\Big(32\,\Big(49z^5+74z^4-52z^3+80z^2-115z+136\Big)\,r^2
$$

$$
-8\,\Big(115z^5+75z^4+9z^3+85z^2-212z+272\Big)\,r+768(z-1)^2z(z+1)r^3+75z^5
$$

$$
+161z^4-37z^3+79z^2-206z+272\Big)+\epsilon\,\Big(16\,\Big(3z^5+86z^4-20z^3+10z^2-79z+96\Big)\,r^2
$$

$$
-8\,\Big(9z^5+89z^4-11z^3-41z^2-46z+96\Big)\,r+1408(z-1)^2z(z+1)r^3-3z^5+124z^4
$$

$$
-34z^3-52z^2-35z+96\Big)-3(z-1)^2z(z+1)\,(1-4r)^2\,(8r-3z+7)\Big)\Bigg]\Bigg\}\,M_4^{\mathrm{R}}
$$

$$
\begin{aligned}
&+\Bigg\{\frac{1}{(\epsilon-1)\epsilon(2\epsilon-1)(4\epsilon-1)\left(1-4r\right)^2(z-1)^2z^2(z+1)}\\
&\Bigg[2r\Bigg(128(z+1)\Big(32(z-1)^4(z+1)\epsilon^5-2\left(13z^5-79z^4+75z^3+31z^2-100z+28\right)\epsilon^4\\
&+\left(7z^5-159z^4+213z^3-125z^2-76z-4\right)\epsilon^3-\left(z^5-102z^4+169z^3-200z^2+72z-52\right)\epsilon^2\\
&+\left(-33z^4+62z^3-89z^2+52z-28\right)\epsilon+4\left(z^2-z+1\right)^2\Big)r^3\\
&-16\Big(8\left(9z^7-11z^6-58z^5+38z^4+57z^3-59z^2-24z+16\right)\epsilon^5\\
&+\left(-34z^7+286z^6+756z^5-508z^4-514z^3+926z^2+656z-224\right)\epsilon^4\\
&+\left(13z^7-437z^6-566z^5+374z^4-143z^3-769z^2-632z-16\right)\epsilon^3\\
&+\left(-3z^7+254z^6+302z^5-172z^4+341z^3+406z^2+152z+208\right)\epsilon^2\\
&-\left(71z^6+84z^5-58z^4+164z^3+99z^2-24z+112\right)\epsilon+8\left(z^2-z+1\right)^2\left(z^2+3z+2\right)\Big)r^2\\
&+2\Big(16\left(19z^7-50z^6-60z^5+94z^4+17z^3-92z^2+8\right)\epsilon^5\\
&-4\left(45z^7-418z^6-248z^5+454z^4-141z^3-484z^2-224z+56\right)\epsilon^4\\
&+2\left(43z^7-1015z^6-4z^5+284z^4-871z^3-269z^2-784z-8\right)\epsilon^3\\
&+\left(-18z^7+1077z^6-126z^5+184z^4+1056z^3+35z^2+848z+208\right)\epsilon^2\\
&-\left(287z^6-70z^5+120z^4+342z^3-55z^2+192z+112\right)\epsilon\\
&+16\left(z^2-z+1\right)^2\left(2z^2+3z+1\right)\Big)r-z\Big(8\left(9z^6-26z^5-26z^4+52z^3-7z^2-42z+8\right)\epsilon^5\\
&+\left(-34z^6+384z^5+188z^4-472z^3+342z^2+344z+80\right)\epsilon^4\\
&+\left(13z^6-468z^5+100z^4+86z^3-513z^2+70z-312\right)\epsilon^3\\
&+\left(-3z^6+259z^5-122z^4+118z^3+229z^2-121z+232\right)\epsilon^2\\
&-\left(71z^5-50z^4+60z^3+58z^2-51z+72\right)\epsilon+8(z+1)\left(z^2-z+1\right)^2\Big)\Bigg)\Bigg]\Bigg\}M_5^{\mathrm{R}}
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg\{\frac{1}{(z-1)^2(\epsilon-1)(2\epsilon-3)\left(1-4r\right)^2}\\
&\Bigg[2r\left(4\epsilon^2\left(16\left(z^2-6z+1\right)r^2-4\left(z^3-8z^2+5z-6\right)r-3z^3+3z^2-z-3\right)\right.\\
&+\epsilon\left(-192\left(z^2-4z+1\right)r^2+8\left(5z^3-19z^2-z-9\right)r+11z^3-13z^2+17z+9\right)\\
&\left.-4(z-1)^2\epsilon^3\left(8r-z-1\right)+(z-1)^2\left(8r+1\right)\left(16r-3z-1\right)\right)\Bigg]\Bigg\}M_6^{\mathrm{R}}
\end{aligned}
$$

$$+\left\{\frac{2r\left(\epsilon\left(-8r+z+1\right)+2r\left(16r-3z-1\right)\right)}{\left(1-4r\right)^2}\right\}M_7^{\mathrm{R}}$$

$$+\Bigg\{-\frac{1}{(z-1)(\epsilon-1)\epsilon(2\epsilon-3)\left(1-4r\right)^2}$$

$$\Bigg[2r\left(-6\left(1-4r\right)^2\left(8\left(z^2-z+1\right)r+z\left(-2z^2+3z-3\right)\right)\right.$$

$$+8\epsilon^4\left(4\left(5z^3-z^2-z+5\right)r^2-2\left(5z^3-2z^2+z+4\right)r+z^3+1\right)$$

$$+\epsilon^3\left(-32\left(20z^3-z^2+12z+25\right)r^2+4\left(83z^3-45z^2+49z+65\right)r\right.$$

$$\left.+384(z+1)^2r^3-35z^3+13z^2-13z-29\right)$$

$$-2\epsilon^2\left(64\left(11z^2+8z+11\right)r^3-8\left(61z^3+5z^2+55z+91\right)r^2\right.$$

$$\left.+\left(259z^3-189z^2+233z+185\right)r-29z^3+23z^2-23z-17\right)$$

$$+\epsilon\left(256\left(7z^2-3z+7\right)r^3-16\left(43z^3+11z^2+15z+75\right)r^2\right.$$

$$+\left(362z^3-326z^2+366z+222\right)r$$

$$\left.\left.-43z^3+51z^2-51z-13\right)\right)\Bigg]\Bigg\}M_8^{\mathrm{R}}$$

$$+\Bigg\{-\frac{1}{\left(1-4r\right)^2}\Bigg[4r\left(\epsilon\left(4\left(3z^2-12z+5\right)r^2\right.\right.$$

$$\left.+\left(-6z^2+22z-8\right)r+z^2-3z+1\right)-2\left(9z^2+2z-11\right)r^2$$

$$\left.+\left(7z^2-10z+3\right)r+96(z-1)r^3-(z-1)^2\right)\Bigg]\Bigg\}M_9^{\mathrm{R}}$$

$$
\begin{aligned}
&+\Bigg\{2r\left(8\left(z^2-z+1\right)r\right.\\
&\quad\left.+z\left(2\left(z^2-2z+2\right)\epsilon-2z^2+3z-3\right)\right)\Bigg\}M^{\mathrm{R}}_{10}\\
&\quad+\Bigg\{\\
&\quad-\frac{1}{(z-1)^2z^2(z+1)(\epsilon-1)\epsilon(4\epsilon-1)\left(1-4r\right)^2}\Bigg[2r\Bigg(256(z\\
&\quad+1)r^3\Bigg(2\left(z^2-z+1\right)^2\\
&\quad+\left(-25z^4+50z^3-29z^2-24z+12\right)\epsilon^3\\
&\quad+2\left(13z^4-26z^3+34z^2-14z+7\right)\epsilon^2\\
&\quad+\left(-13z^4+26z^3-39z^2+24z-12\right)\epsilon+16(z-1)^3(z+1)\epsilon^4\Bigg)\\
&\quad-32r^2\Bigg(4\left(z^2-z+1\right)^2\left(z^2+3z+2\right)\\
&\quad+4\left(13z^6+16z^5-42z^4-16z^3+53z^2+8z-16\right)\epsilon^4\\
&\quad-\left(71z^6+112z^5-182z^4+32z^3+263z^2+56z-48\right)\epsilon^3\\
&\quad+\left(68z^6+73z^5-87z^4+171z^3+99z^2+12z+56\right)\epsilon^2\\
&\quad-\left(29z^6+29z^5-29z^4+87z^3+28z^2-16z+48\right)\epsilon\Bigg)\\
&\quad+2r\Bigg(16\left(z^2-z+1\right)^2\left(2z^2+3z+1\right)\\
&\quad+16\left(27z^6+4z^5-56z^4+16z^3+57z^2-8z-8\right)\epsilon^4\\
&\quad-8\left(78z^6+11z^5-82z^4+95z^3+98z^2+20z-12\right)\epsilon^3\\
&\quad+\left(593z^6-216z^5+94z^4+984z^3-207z^2+432z+112\right)\epsilon^2\\
&\quad-\left(241z^6-112z^5+110z^4+384z^3-143z^2+160z+96\right)\epsilon\Bigg)\\
&\quad+z\Bigg(-8(z+1)\left(z^2-z+1\right)^2\\
&\quad-4\left(25z^5-50z^3+32z^2+41z-16\right)\epsilon^4\\
&\quad+\left(137z^5+8z^4-130z^3+232z^2+89z+16\right)\epsilon^3\\
&\quad-\left(135z^5-82z^4+66z^3+214z^2-133z+136\right)\epsilon^2\\
&\quad+\left(58z^5-50z^4+52z^3+70z^2-66z+64\right)\epsilon\Bigg)\Bigg)\Bigg]\Bigg\}M^{\mathrm{R}}_{11}
\end{aligned}
$$

$$+\Bigg\{-\frac{1}{z(z+1)(\epsilon-1)(2\epsilon-1)(4\epsilon-1)\,(1-4r)^2}\Bigg[2r\left(z\left(\left(-48z^2+64z+304\right)r^2\right.\right.$$

$$\left.+4\left(9z^2-19z-44\right)r+128(z+1)r^3-3z^2+10z+23\right)$$

$$+8\epsilon^3\left(16\left(9z^3+16z^2-9z-16\right)r^2-4\left(19z^3+31z^2-18z-32\right)r+9z^3+16z^2-9z-16\right)$$

$$+2\epsilon^2\left(-16\left(17z^3+40z^2-81z-80\right)r^2+4\left(45z^3+59z^2-168z-160\right)r\right.$$

$$\left.+128z(z+1)r^3-17z^3-32z^2+85z+80\right)-\epsilon\left(16\left(-13z^3+4z^2+109z+32\right)r^2\right.$$

$$\left.\left.+4\left(43z^3-53z^2-236z-64\right)r+384z(z+1)r^3-13z^3+26z^2+121z+32\right)\right)\Bigg]\Bigg\}\,M_{12}^{\mathrm{R}}$$

$$
\begin{aligned}
&+\Bigg\{-\frac{1}{(\epsilon-1)\epsilon(2\epsilon-1)(4\epsilon-1)\left(1-4r\right)^2\left(4r-z+1\right)(z-1)^2z^2} \\
&\Bigg[256(3\epsilon-2)\left(64(z-1)^3(z+1)\epsilon^4-4\left(25z^4-50z^3+29z^2+24z-12\right)\epsilon^3\right. \\
&+2\left(53z^4-106z^3+137z^2-56z+28\right)\epsilon^2+\left(-55z^4+110z^3-159z^2+96z-48\right)\epsilon \\
&\left.+9z^4-18z^3+25z^2-16z+8\right)r^4-32\left(48\left(13z^5-13z^4-5z^3+5z^2+16z-8\right)\epsilon^5\right. \\
&-4\left(407z^5-455z^4+257z^3-17z^2+464z-136\right)\epsilon^4 \\
&+2\left(1001z^5-1251z^4+1515z^3-337z^2+784z+72\right)\epsilon^3 \\
&-\left(1339z^5-1809z^4+2617z^3-579z^2+544z+512\right)\epsilon^2 \\
&\left.+\left(443z^5-633z^4+921z^3-187z^2+64z+240\right)\epsilon-54z^5+82z^4-114z^3+22z^2-32\right)r^3 \\
&+8\left(48\left(2z^6+24z^5-45z^4+22z^3+29z^2-24z+8\right)\epsilon^5\right. \\
&-4\left(100z^6+591z^5-1038z^4+875z^3+304z^2-216z+136\right)\epsilon^4 \\
&+\left(614z^6+2207z^5-4121z^4+6373z^3-2865z^2+2288z-144\right)\epsilon^3 \\
&+\left(-434z^6-1251z^5+2823z^4-5661z^3+4219z^2-3104z+512\right)\epsilon^2 \\
&+2\left(70z^6+202z^5-547z^4+1116z^3-913z^2+632z-120\right)\epsilon \\
&\left.-4\left(4z^6+13z^5-40z^4+77z^3-62z^2+40z-8\right)\right)r^2-2\left(-32z^6+58z^5-70z^4+14z^3\right. \\
&+30z^2-64z+\epsilon^2\left(-829z^6+1285z^5-2349z^4+2199z^3-1394z^2-480z+512\right) \\
&+24\epsilon^5\left(8z^6+17z^5-33z^4-17z^3+89z^2-64z+16\right) \\
&+\epsilon\left(274z^6-485z^5+735z^4-431z^3+35z^2+416z-240\right) \\
&-2\epsilon^4\left(382z^6+7z^5+189z^4-1447z^3+2725z^2-1472z+272\right) \\
&\left.+\epsilon^3\left(1159z^6-1204z^5+2710z^4-4124z^3+4595z^2-1280z-144\right)+32\right)r \\
&+\left(12\epsilon^3-23\epsilon^2+13\epsilon-2\right)(z-1)z\left(\left(5z^4-5z^3+19z^2-27z+16\right)\epsilon^2\right. \\
&\left.+\left(-9z^4+19z^3-43z^2+45z-24\right)\epsilon+2\left(2z^4-5z^3+9z^2-8z+4\right)\right)\Bigg]\Bigg\}M_{13}^{\mathrm{R}}
\end{aligned}
$$

$$+\left\{\frac{1}{(4\epsilon-1)(2r-z-1)}\left[r\left(8(z-1)\epsilon^2\left(-2\left(z^2+1\right)r+2(z+3)r^2+z^3\right.\right.\right.\right.$$
$$\left.-z^2-3z-1\right)-(r-1)\left(\left(5z^2+2z-23\right)r+8(z-1)r^2-z^3-4z^2+z+4\right)$$
$$+2\epsilon\left(-8\left(z^2+4z+3\right)r^2+4\left(3z^3-10z^2+z+6\right)r\right.$$
$$\left.\left.\left.\left.+16(z+3)r^3-3\left(z^4-4z^3-2z^2+4z+1\right)\right)\right)\right]\right\}M_{14}^{\mathrm{R}}$$
$$+\left\{-\frac{2(z-1)(2\epsilon+1)r^2(8r+4(z-1)\epsilon-3z+7)}{4\epsilon-1}\right\}M_{15}^{\mathrm{R}}$$
$$+\left\{-\frac{1}{(4\epsilon-1)(2r-z-1)}\right.$$
$$\left[2r\left(\left(-11z^2+4z+15\right)r\right.\right.$$
$$+4\epsilon^2\left(\left(-6z^2+4z+2\right)r+4(z-1)r^2+3z^3+z^2-3z-1\right)$$
$$+\epsilon\left(\left(62z^2-36z-58\right)r+(16-64z)r^2+32r^3-13z^3+5z^2+21z+3\right)$$
$$\left.\left.\left.+(11z+1)r^2-8r^3+2\left(z^3-2z-1\right)\right)\right]\right\}M_{16}^{\mathrm{R}}$$

A.3 Matrix Elements for Quark Channels

$$A_{q\bar{q}}=\left\{\frac{8(\epsilon-1)r((z-1)\epsilon+1)}{2\epsilon-3}\right\}M_4^{\mathrm{R}}+\left\{-\frac{8(\epsilon-1)r}{2\epsilon-3}\right\}M_6^{\mathrm{R}}$$
$$+\left\{\frac{4(z-1)(\epsilon-1)r\left(4r+(z-1)\epsilon-z+1\right)}{2\epsilon-3}\right\}M_8^{\mathrm{R}}$$

$$
\begin{aligned}
A_{qg} = {} & \left\{ \frac{8((z+2)\epsilon - z)}{z(z+1)(2\epsilon-1)} \right\} M_1^{\mathrm{R}} \\
& + \left\{ \vphantom{\frac{1}{1}} \right. \\
& - \frac{4r\left(z^3 - \left(3z^2+z+2\right) z\epsilon - z^2 + 2\left(z^3+3z^2-z-1\right)\epsilon^2 + 2\right) + z(z+1)(\epsilon-1)^2\left(z^2(\epsilon-1)+2z-2\right)}{z^2(z+1)(\epsilon-1)\epsilon} \left. \vphantom{\frac{1}{1}} \right\} \\
& M_2^{\mathrm{R}} + \left\{ \frac{4r\left(z^3(\epsilon-1)\epsilon + z^2\left(2\epsilon^2+\epsilon+1\right) + z\left(\epsilon^2-6\epsilon-3\right)+4\right)}{(z-1)z(z+1)(\epsilon-1)} \right\} M_4^{\mathrm{R}} \\
& + \left\{ \frac{1}{(z-1)z^2(z+1)(\epsilon-1)\epsilon(2\epsilon-1)} \Big[2r\left(4(z+1)\left(2\epsilon^2-3\epsilon+1\right) r \left(-z^2\right.\right.\right. \\
& \qquad \left. + (4z-2)\epsilon + 2z - 2\right) + z\left(z^3 - z^2 - \left(3z^3+8z^2+17z-4\right)\epsilon^3\right. \\
& \qquad \left.\left. + \left(5z^3+11z^2+10\right)\epsilon^2 - \left(5z^3-2z^2+z+8\right)\epsilon + 2z(z+1)^2\epsilon^4 + 2\right)\right) \Big] \Big\} M_5^{\mathrm{R}} \\
& + \left\{ \frac{1}{(z-1)z^2(z+1)(\epsilon-1)\epsilon} \Big[2r\left(4(z+1)(2\epsilon-1) r \left(\left(z^2+2z-2\right)\epsilon - z^2 + 2z - 2\right)\right.\right. \\
& \qquad + z\left(-z^3 + z^2 - \left(3z^3+5z^2+6z-4\right)\epsilon^2 + \left(3z^3+2z^2-3z+6\right)\epsilon + z(z+1)^2\epsilon^3\right. \\
& \qquad \left.\left. - 2\right)\right) \Big] \Big\} M_{11}^{\mathrm{R}} \\
& + \left\{ \frac{8r((z+2)\epsilon - z)}{z(z+1)(\epsilon-1)(2\epsilon-1)} \right\} M_{12}^{\mathrm{R}} \\
& + \left\{ \frac{(3\epsilon-2)\left(4r\left(\left(z^2+2z-2\right)\epsilon - z^2 + 2z - 2\right) + z(\epsilon-1)\left(z^2(\epsilon-1)+2z-2\right)\right)}{(z-1)z^2(\epsilon-1)\epsilon} \right\} M_{13}^{\mathrm{R}}
\end{aligned}
$$

Appendix B

Solutions to Differential Equations

B.1 Double-Virtual Contribution

$$M_1^{\mathrm{V}} = x^{2-2\epsilon}\, BC_1^{\mathrm{V}} \tag{B.1}$$

$$M_2^{\mathrm{V}} = x^{1-\epsilon}\, BC_2^{\mathrm{V}} \tag{B.2}$$

$$M_3^{\mathrm{V}} = -x^{2-\epsilon}\,(4-6\epsilon)BC_3^{\mathrm{V}} - x^{2-2\epsilon}\,2BC_1^{\mathrm{V}} \tag{B.3}$$

$$M_4^{\mathrm{V}} = \frac{BC_4^{\mathrm{V}}\left(x^2(2-4\epsilon)+x\right) - 2BC_2^{\mathrm{V}}x^{2-\epsilon}}{x} + \mathcal{O}(x^2) \tag{B.4}$$

$$\begin{aligned}
M_5^{\mathrm{V}} = {} & \frac{(\epsilon-1)^2 BC_1^{\mathrm{V}}\left(12x\epsilon^2+(6x-1)\epsilon-1\right)}{\epsilon+1} \\
& - \frac{BC_5^{\mathrm{V}}\left(-12x\epsilon^2-6x\epsilon+\epsilon+1\right)}{\epsilon+1} + \frac{2BC_6^{\mathrm{V}}\left(-12x\epsilon^2-6x\epsilon+\epsilon+1\right)}{\epsilon+1} \\
& + x^{-\epsilon}\Big(-2(\epsilon-1)BC_1^{\mathrm{V}}\left(2x\epsilon^2-(6x+1)\epsilon+1\right) \\
& + \frac{2BC_5^{\mathrm{V}}\left(-2x\epsilon^2+6x\epsilon+\epsilon-1\right)}{\epsilon-1} + \frac{BC_6^{\mathrm{V}}\left(4x\epsilon^2-2(6x+1)\epsilon+2\right)}{\epsilon-1}\Big) \\
& + x^{-2\epsilon}\Big((\epsilon-1)BC_1^{\mathrm{V}}\left(4x\epsilon^2-(6x+1)\epsilon+1\right)\Big) \\
& + \mathcal{O}(x^2)
\end{aligned} \tag{B.5}$$

$$
\begin{aligned}
M_6^{\mathrm{V}} = & \frac{(\epsilon-1)^2 BC_1^{\mathrm{V}}\left(12x\epsilon^2+(12x-1)\epsilon+3x-1\right)}{\epsilon+1} \\
& + \frac{BC_5^{\mathrm{V}}\left(12x\epsilon^2+(12x-1)\epsilon+3x-1\right)}{\epsilon+1} + \frac{BC_6^{\mathrm{V}}\left(-24x\epsilon^2-24x\epsilon-6x+2\epsilon+2\right)}{\epsilon+1} \\
& + x^{-\epsilon}\bigg(-(\epsilon-1)BC_1^{\mathrm{V}}\left(2x\epsilon^2-(6x+1)\epsilon-4x+1\right) \\
& + \frac{BC_5^{\mathrm{V}}\left(-2x\epsilon^2+6x\epsilon+4x+\epsilon-1\right)}{\epsilon-1} + \frac{BC_6^{\mathrm{V}}\left(2x\epsilon^2-(6x+1)\epsilon-4x+1\right)}{\epsilon-1}\bigg) \\
& + x^{-2\epsilon+1}\left(BC_1^{\mathrm{V}}(1-\epsilon)\right) \\
& + \mathcal{O}(x^2)
\end{aligned}
\tag{B.6}
$$

$$
\begin{aligned}
M_7^{\mathrm{V}} = & -\frac{(3\epsilon-2)BC_{12}^{\mathrm{V}}\left(6x\epsilon^2+(x-2)\epsilon-x\right)}{\epsilon(2\epsilon-1)} \\
& + x^{-\epsilon}\left(\frac{x(2\epsilon-1)BC_3^{\mathrm{V}}}{\epsilon}\right) \\
& + x^{-2\epsilon}\bigg(-\frac{x(6\epsilon-1)(\epsilon-1)^2 BC_1^{\mathrm{V}}\log(x)}{8\epsilon^2-6\epsilon+1} \\
& + \frac{2xBC_8^{\mathrm{V}}\log(x)}{4\epsilon-1} - \frac{xBC_7^{\mathrm{V}}(2(2\epsilon-1)\epsilon\log(x)-4\epsilon+1)}{4\epsilon-1} \\
& + \frac{x\left(3\epsilon^2-5\epsilon+2\right)BC_{12}^{\mathrm{V}}(2(2\epsilon-1)\epsilon\log(x)-4\epsilon+1)}{(1-2\epsilon)^2\epsilon(4\epsilon-1)} \\
& + \frac{x(2\epsilon-1)BC_3^{\mathrm{V}}((3\epsilon-1)\epsilon\log(x)-4\epsilon+1)}{\epsilon(4\epsilon-1)}\bigg) \\
& + \mathcal{O}(x^2)
\end{aligned}
\tag{B.7}
$$

$$
\begin{aligned}
M_8^{\mathrm{V}} = {} & \frac{3\epsilon\left(9\epsilon^2-9\epsilon+2\right)BC_{12}^{\mathrm{V}}}{2\epsilon+1} \\
& + x^{-\epsilon}\Bigg(-\frac{\left(2\epsilon^2-3\epsilon+1\right)BC_3^{\mathrm{V}}\left(6x\epsilon^2+(10x-1)\epsilon+2x-1\right)}{2x(\epsilon+1)}\Bigg) \\
& + x^{-2\epsilon}\Bigg(-\frac{(1-2\epsilon)^2\epsilon^2 BC_7^{\mathrm{V}}}{x(4\epsilon-1)}\left(x\left(-8\epsilon^2+2\epsilon+3\right)+2\left(x\left(2\epsilon^2-5\epsilon-2\right)+1\right)\log(x)\right) \\
& + \frac{\epsilon\left(3\epsilon^2-5\epsilon+2\right)BC_{12}^{\mathrm{V}}}{x(4\epsilon-1)}\left(x\left(-8\epsilon^2+2\epsilon+3\right)+2\left(x\left(2\epsilon^2-5\epsilon-2\right)+1\right)\log(x)\right) \\
& + \frac{BC_8^{\mathrm{V}}}{x(4\epsilon-1)}\left(-16x\epsilon^4+20x\epsilon^3-18x\epsilon^2+(4-6x)\epsilon\right. \\
& \qquad \left. +2(2\epsilon-1)\epsilon\left(x\left(2\epsilon^2-5\epsilon-2\right)+1\right)\log(x)+2x-1\right)+\frac{BC_1^{\mathrm{V}}}{x(4\epsilon-1)} \\
& -(\epsilon-1)\epsilon\left(x\epsilon\left(-24\epsilon^3+30\epsilon^2+\epsilon-4\right)+\left(6\epsilon^2-7\epsilon+1\right)\left(x\left(2\epsilon^2-5\epsilon-2\right)+1\right)\log(x)\right) \\
& + \frac{BC_3^{\mathrm{V}}}{2x(4\epsilon-1)}\Bigg((2\epsilon-1)\left(-48x\epsilon^5+44x\epsilon^4+30x\epsilon^3-4(8x+1)\epsilon^2+(5-2x)\epsilon\right. \\
& \qquad \left. +2\left(6\epsilon^2-5\epsilon+1\right)\epsilon\left(x\left(2\epsilon^2-5\epsilon-2\right)+1\right)\log(x)+2x-1\right)\Bigg)\Bigg) \\
& + \mathcal{O}(x^2)
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
M_9^{\mathrm{V}} = &\frac{(\epsilon-1)^2 BC_1^{\mathrm{V}}\left(12x\epsilon^3+(6x-1)\epsilon^2-2\epsilon-1\right)}{\epsilon(\epsilon+1)^2(2\epsilon-1)} \\
&+\frac{BC_4^{\mathrm{V}}\left(-8x\epsilon^3+2x\epsilon+2\epsilon^2+\epsilon-1\right)}{\epsilon(\epsilon+1)}+\frac{BC_5^{\mathrm{V}}\left(-12x\epsilon^3-6x\epsilon^2+\epsilon^2+2\epsilon+1\right)}{-2\epsilon^4-3\epsilon^3+\epsilon} \\
&+\frac{2BC_6^{\mathrm{V}}\left(-12x\epsilon^3+(1-6x)\epsilon^2+2\epsilon+1\right)}{\epsilon(\epsilon+1)^2(2\epsilon-1)} \\
&+x^{-\epsilon}\Bigg(BC_4^{\mathrm{V}}\left(4x\epsilon-2(x+1)+\frac{1}{\epsilon}\right)+BC_9^{\mathrm{V}}(1-2x\epsilon) \\
&+(\epsilon-1)BC_3^{\mathrm{V}}(4x\epsilon+(2x\epsilon-1)\log(x))+BC_2^{\mathrm{V}}((\epsilon-1)(2x\epsilon-1)\log(x)-2x\epsilon) \\
&+\frac{BC_5^{\mathrm{V}}\left(10x\epsilon^2-2x\epsilon-2(\epsilon-1)\epsilon(2x\epsilon-1)\log(x)+\epsilon-1\right)}{(\epsilon-1)\epsilon(2\epsilon-1)} \\
&-\frac{(\epsilon-1)BC_1^{\mathrm{V}}\left(-10x\epsilon^2+(2x-1)\epsilon+2(\epsilon-1)\epsilon(2x\epsilon-1)\log(x)+1\right)}{\epsilon(2\epsilon-1)} \\
&+\frac{BC_6^{\mathrm{V}}\left(2(\epsilon-1)\epsilon(2x\epsilon-1)\log(x)-2\left(4x\epsilon^2+\epsilon-1\right)\right)}{(\epsilon-1)\epsilon(2\epsilon-1)}\Bigg) \\
&+x^{-2\epsilon}\left(\frac{2x(\epsilon-1)BC_1^{\mathrm{V}}}{2\epsilon-1}\right) \\
&+\mathcal{O}(x^2)
\end{aligned}
\tag{B.9}
$$

$$
\begin{aligned}
M_{10}^{\mathrm{V}} = &\; x^{-\epsilon}\left(\frac{x(2\epsilon-1)BC_3^{\mathrm{V}}}{\epsilon}\right) \\
&x^{-2\epsilon}\left(x\left(\frac{1}{\epsilon}-2\right)BC_3^{\mathrm{V}}-x(\epsilon-1)BC_1^{\mathrm{V}}\log(x)+xBC_{10}^{\mathrm{V}}\right) \\
&+\mathcal{O}(x^2)
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
M_{11}^{\mathrm{V}} = {} & \frac{(\epsilon-1)^2 BC_1^{\mathrm{V}}\Big(12x\epsilon^3-(4x+1)\epsilon^2-(3x+1)\epsilon+x\Big)}{2\epsilon(\epsilon+1)\left(6\epsilon^2-5\epsilon+1\right)} \\
& + BC_5^{\mathrm{V}}\left(\frac{2x\epsilon+x}{2\epsilon^2+2\epsilon}-\frac{1}{2\left(6\epsilon^2-5\epsilon+1\right)}\right)+BC_6^{\mathrm{V}}\left(\frac{1}{6\epsilon^2-5\epsilon+1}-\frac{2x\epsilon+x}{\epsilon^2+\epsilon}\right) \\
& x^{-\epsilon}\left(-\frac{(\epsilon-1)BC_1^{\mathrm{V}}\left(2x\epsilon^2-(4x+1)\epsilon+2x\right)}{\epsilon(2\epsilon-1)}\right. \\
& \left.+\frac{BC_5^{\mathrm{V}}\left(-2x\epsilon^2+4x\epsilon-2x+\epsilon\right)}{2\epsilon^3-3\epsilon^2+\epsilon}-\frac{BC_6^{\mathrm{V}}\left(-2x\epsilon^2+4x\epsilon-2x+\epsilon\right)}{2\epsilon^3-3\epsilon^2+\epsilon}\right) \\
& x^{-2\epsilon}\left(-\frac{x\left(19\epsilon^2-16\epsilon+3\right)BC_5^{\mathrm{V}}}{2(\epsilon-1)\epsilon\left(6\epsilon^2-5\epsilon+1\right)}+\frac{x\left(16\epsilon^2-5\epsilon+1\right)BC_6^{\mathrm{V}}}{(\epsilon-1)\epsilon\left(6\epsilon^2-5\epsilon+1\right)}\right. \\
& \left.-\frac{xBC_1^{\mathrm{V}}\left(2\left(6\epsilon^3-11\epsilon^2+6\epsilon-1\right)\epsilon\log(x)+9\epsilon^3-17\epsilon^2+15\epsilon-3\right)}{2\epsilon\left(6\epsilon^2-5\epsilon+1\right)}+xBC_{11}^{\mathrm{V}}\right) \\
& +\mathcal{O}(x^2)
\end{aligned}
\tag{B.11}
$$

$$
M_{12}^{\mathrm{V}} = BC_{12}^{\mathrm{V}}\left(\frac{2x(2-3\epsilon)\epsilon}{2\epsilon-1}+1\right)+\mathcal{O}(x^2) \tag{B.12}
$$

$$
M_{13}^{\mathrm{V}} = \frac{BC_{13}^{\mathrm{V}}\left(x^2(4-8\epsilon)+x\right)-4x^2BC_3^{\mathrm{V}}x^{-\epsilon}}{x}+\mathcal{O}(x^2) \tag{B.13}
$$

$$
\begin{aligned}
M_{14}^{\mathrm{V}} = {} & \frac{3(\epsilon-1)^2BC_1^{\mathrm{V}}}{2\epsilon^2(\epsilon+1)^2}\Big(12x\epsilon^3+(16x-1)\epsilon^2+2(5x-1)\epsilon+2x-1\Big) \\
& +\frac{BC_5^{\mathrm{V}}}{2\epsilon^2(\epsilon+1)^2}\Big(36x\epsilon^3+(48x-3)\epsilon^2+6(5x-1)\epsilon+6x-3\Big) \\
& +\frac{BC_6^{\mathrm{V}}\Big(-36x\epsilon^3+(3-48x)\epsilon^2+(6-30x)\epsilon-6x+3\Big)}{\epsilon^2(\epsilon+1)^2} \\
& -\frac{3BC_{12}^{\mathrm{V}}\left(9\epsilon^2-9\epsilon+2\right)}{\epsilon^2(\epsilon+1)(2\epsilon+1)}\Big(6x\epsilon^3+(20x-2)\epsilon^2+3(4x-1)\epsilon \\
& +2x-1\Big)+\frac{BC_{13}^{\mathrm{V}}\Big(-32x\epsilon^4+16x\epsilon^2-2x+4\epsilon^3-3\epsilon+1\Big)}{\epsilon^3+\epsilon^2}
\end{aligned}
$$

$$
\begin{aligned}
&+x^{-\epsilon}\Bigg(\frac{(1-2\epsilon)^2 BC^{\mathrm{V}}_7(x(6\epsilon+2)-1)}{\epsilon(4\epsilon-1)}\\
&+\frac{BC^{\mathrm{V}}_8(2-4x(3\epsilon+1))}{\epsilon-4\epsilon^2}-\frac{(\epsilon-1)BC^{\mathrm{V}}_{10}(x(6\epsilon+2)-1)}{\epsilon}\\
&+\frac{(\epsilon-1)BC^{\mathrm{V}}_{11}(x(6\epsilon+2)-1)}{\epsilon}+\frac{2\,BC^{\mathrm{V}}_{12}}{\epsilon^2(4\epsilon-1)}\Big(54\epsilon^3\\
&-69\epsilon^2+28\epsilon-4\Big)(x(6\epsilon+2)-1)\\
&+\frac{(1-2\epsilon)^2 BC^{\mathrm{V}}_{13}(x(6\epsilon+2)-1)}{\epsilon^2}+BC^{\mathrm{V}}_{14}(1-2x(3\epsilon+1))\\
&+\frac{BC^{\mathrm{V}}_3(2\epsilon-1)}{\epsilon(\epsilon+1)(4\epsilon-1)}\Big(\epsilon\left(2x\epsilon^2-26x\epsilon+2x+\epsilon+1\right)\\
&+3\left(4\epsilon^3-\epsilon^2-4\epsilon+1\right)(x(6\epsilon+2)-1)\log(x)\Big)\\
&+\frac{BC^{\mathrm{V}}_5}{2(\epsilon-1)\epsilon^2(\epsilon+1)(2\epsilon-1)(3\epsilon-1)}\Big(96x\epsilon^6-206x\epsilon^5\\
&+(144x+37)\epsilon^4+(184x-31)\epsilon^3-(108x+31)\epsilon^2+(31-26x)\epsilon\\
&-4\left(6\epsilon^4-5\epsilon^3-5\epsilon^2+5\epsilon-1\right)\epsilon(x(6\epsilon+2)-1)\log(x)+12x-6\Big)\\
&+\frac{2\,BC^{\mathrm{V}}_6}{(\epsilon-1)\epsilon^2(\epsilon+1)(2\epsilon-1)(3\epsilon-1)}\Big(-24x\epsilon^6+98x\epsilon^5\\
&-17(2x+1)\epsilon^4+(10-94x)\epsilon^3+(26x+15)\epsilon^2+2(4x-5)\epsilon\\
&+\left(6\epsilon^4-5\epsilon^3-5\epsilon^2+5\epsilon-1\right)\epsilon(x(6\epsilon+2)-1)\log(x)-4x+2\Big)\\
&-\frac{(\epsilon-1)BC^{\mathrm{V}}_1}{2\epsilon^2(\epsilon+1)(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)}\Big(-384x\epsilon^7\\
&+464x\epsilon^6-2(341x+36)\epsilon^5+(119-88x)\epsilon^4\\
&+(504x+23)\epsilon^3-(52x+113)\epsilon^2+(49-62x)\epsilon\\
&+4\left(24\epsilon^5-26\epsilon^4-15\epsilon^3+25\epsilon^2-9\epsilon+1\right)\epsilon(x(6\epsilon+2)-1)\log(x)\\
&+12x-6\Big)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+x^{-2\epsilon}\Bigg(\frac{\left(19\epsilon^2-16\epsilon+3\right)BC_5^{\mathrm{V}}\left(4x\epsilon^2-(4x+1)\epsilon-2x+1\right)}{2(\epsilon-1)\epsilon^2(2\epsilon-1)(3\epsilon-1)} \qquad \text{(B.14)}\\
&-\frac{\left(16\epsilon^2-5\epsilon+1\right)BC_6^{\mathrm{V}}\left(4x\epsilon^2-(4x+1)\epsilon-2x+1\right)}{(\epsilon-1)\epsilon^2(2\epsilon-1)(3\epsilon-1)}\\
&+BC_{10}^{\mathrm{V}}\left(\frac{1-2x}{\epsilon}+4x\epsilon-4x-1\right)\\
&+\frac{BC_{11}^{\mathrm{V}}\left(-4x\epsilon^2+4x\epsilon+2x+\epsilon-1\right)}{\epsilon}+\frac{2\,BC_8^{\mathrm{V}}}{(\epsilon-1)\epsilon(4\epsilon-1)}\\
&\Big(-8x\epsilon^4+20x\epsilon^3-20x\epsilon^2+8x\epsilon\\
&+\left(2\epsilon^2-3\epsilon+1\right)\left(2x\left(\epsilon^2-3\epsilon-1\right)+1\right)\log(x)+2x+\epsilon-1\Big)\\
&-\frac{\left(4\epsilon^3BC_7^{\mathrm{V}}-\epsilon^2\left(4BC_7^{\mathrm{V}}+3BC_{12}^{\mathrm{V}}\right)+\epsilon\left(BC_7^{\mathrm{V}}+5BC_{12}^{\mathrm{V}}\right)-2BC_{12}^{\mathrm{V}}\right)}{(\epsilon-1)\epsilon^2(2\epsilon-1)(4\epsilon-1)}\\
&\Bigg(-16x\epsilon^5+32x\epsilon^4-6x\epsilon^3-2(4x+1)\epsilon^2\\
&+2\left(2\epsilon^2-3\epsilon+1\right)\epsilon\left(2x\left(\epsilon^2-3\epsilon-1\right)+1\right)\log(x)+2x+3\epsilon-1\Bigg)\\
&+\frac{(2\epsilon-1)BC_3^{\mathrm{V}}}{(\epsilon-1)\epsilon(4\epsilon-1)}\Big(\left(6\epsilon^3-11\epsilon^2+6\epsilon-1\right)\left(2x\left(\epsilon^2-3\epsilon-1\right)+1\right)\log(x)\\
&-\epsilon\left(24x\epsilon^4-52x\epsilon^3+20x\epsilon^2+2x\epsilon+2x+\epsilon-1\right)\Big)\\
&+\frac{BC_1^{\mathrm{V}}}{2\epsilon^2(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)}\Big(288x\epsilon^8-960x\epsilon^7+1032x\epsilon^6-700x\epsilon^5\\
&+(560x+41)\epsilon^4-26(11x+4)\epsilon^3+(88-14x)\epsilon^2+(38x-28)\epsilon\\
&-2(\epsilon-1)^2\left(36\epsilon^3-36\epsilon^2+11\epsilon-1\right)\epsilon\left(2x\left(\epsilon^2-3\epsilon-1\right)+1\right)\log(x)\\
&-6x+3\Big)\Bigg)+\mathcal{O}(x^2)
\end{aligned}
$$

$$M_{15}^{\mathrm{V}} = -\frac{(\epsilon-1)^2 BC_1^{\mathrm{V}}}{4\epsilon^2(\epsilon+1)^2(2\epsilon+1)}\Big(24x\epsilon^4+(36x-2)\epsilon^3+5(6x-1)\epsilon^2+(6x-4)\epsilon-1\Big)$$
$$+\frac{BC_5^{\mathrm{V}}\Big(-24x\epsilon^4+(2-36x)\epsilon^3+(5-30x)\epsilon^2+(4-6x)\epsilon+1\Big)}{4\epsilon^2(\epsilon+1)^2(2\epsilon+1)}$$
$$+\frac{BC_6^{\mathrm{V}}\Big(24x\epsilon^4+(36x-2)\epsilon^3+5(6x-1)\epsilon^2+(6x-4)\epsilon-1\Big)}{2\epsilon^2(\epsilon+1)^2(2\epsilon+1)}$$
$$+\frac{\Big(9\epsilon^2-9\epsilon+2\Big)BC_{12}^{\mathrm{V}}\Big(6x\epsilon^3+(15x-2)\epsilon^2+3(x-1)\epsilon-1\Big)}{\epsilon^2(\epsilon+1)(2\epsilon+1)}$$

$$+x^{-\epsilon}\Bigg(\Big(6x\epsilon^2+(4x-1)\epsilon-1\Big)$$
$$\Bigg(-\frac{(1-2\epsilon)^2BC_7^{\mathrm{V}}}{2\epsilon(\epsilon+1)(4\epsilon-1)}-\frac{BC_8^{\mathrm{V}}}{\epsilon\left(4\epsilon^2+3\epsilon-1\right)}+\frac{(\epsilon-1)BC_{10}^{\mathrm{V}}}{2\epsilon(\epsilon+1)}-\frac{(\epsilon-1)BC_{11}^{\mathrm{V}}}{2\epsilon(\epsilon+1)}\Bigg)$$
$$-\frac{BC_{12}^{\mathrm{V}}\Big(54\epsilon^3-69\epsilon^2+28\epsilon-4\Big)\Big(6x\epsilon^2+(4x-1)\epsilon-1\Big)}{\epsilon^2(\epsilon+1)(4\epsilon-1)}$$
$$-\frac{BC_{14}^{\mathrm{V}}\Big(-6x\epsilon^2-4x\epsilon+\epsilon+1\Big)}{2\epsilon+2}-\frac{(2\epsilon-1)BC_{16}^{\mathrm{V}}\Big(6x\epsilon^2+(4x-1)\epsilon-1\Big)}{2\epsilon(\epsilon+1)}$$
$$-\frac{(2\epsilon-1)BC_3^{\mathrm{V}}}{2\epsilon^2(\epsilon+1)^2(4\epsilon-1)}\Big(-38x\epsilon^5+(9-10x)\epsilon^4+(30x+8)\epsilon^3+(32x-9)\epsilon^2$$
$$-2(5x+3)\epsilon+2\left(4\epsilon^3-\epsilon^2-4\epsilon+1\right)\epsilon\left(6x\epsilon^2+(4x-1)\epsilon-1\right)\log(x)+2\Big)$$
$$+\frac{BC_6^{\mathrm{V}}}{(\epsilon-1)\epsilon(\epsilon+1)^2(2\epsilon-1)(3\epsilon-1)}\Big(24x\epsilon^6-50x\epsilon^5+(5-40x)\epsilon^4+(86x+5)\epsilon^3$$
$$-(2x+5)\epsilon^2-(4x+5)\epsilon-\Big(6\epsilon^4-5\epsilon^3-5\epsilon^2+5\epsilon-1\Big)$$
$$\Big(6x\epsilon^2+(4x-1)\epsilon-1\Big)\log(x)+2x\Big)$$
$$+\frac{BC_5^{\mathrm{V}}}{4(\epsilon-1)\epsilon^2(\epsilon+1)^2(2\epsilon-1)(3\epsilon-1)}\Big(-96x\epsilon^7+14x\epsilon^6+(90x+11)\epsilon^5+(2-134x)\epsilon^4$$
$$+2(35x-9)\epsilon^3-8x\epsilon^2+4\left(6\epsilon^4-5\epsilon^3-5\epsilon^2+5\epsilon-1\right)\epsilon$$
$$\Big(6x\epsilon^2+(4x-1)\epsilon-1\Big)\log(x)+7\epsilon-2\Big)$$
$$+\frac{(\epsilon-1)BC_1^{\mathrm{V}}}{4\epsilon^2(\epsilon+1)^2(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)}\Bigg(-384x\epsilon^8-304x\epsilon^7+6(49x+20)\epsilon^6+(31-38x)\epsilon^5$$

$$+2(221x-93)\epsilon^4-2(117x+5)\epsilon^3+32(x+2)\epsilon^2+4\left(24\epsilon^5-26\epsilon^4-15\epsilon^3+25\epsilon^2-9\epsilon+1\right)$$
$$\epsilon\left(6x\epsilon^2+(4x-1)\epsilon-1\right)\log(x)-21\epsilon+2\Bigg)\Bigg)$$

$$+x^{-2\epsilon}\Bigg(\frac{BC_5^{\mathrm{V}}\left(68x\epsilon^4-(86x+17)\epsilon^3+(36x+31)\epsilon^2-(6x+17)\epsilon+3\right)}{4(\epsilon-1)\epsilon^2(2\epsilon-1)(3\epsilon-1)}$$
$$-\frac{BC_6^{\mathrm{V}}\left(-16x\epsilon^4+4(7x+1)\epsilon^3+(14x+1)\epsilon^2-2(x+3)\epsilon+1\right)}{2\epsilon^2\left(6\epsilon^3-11\epsilon^2+6\epsilon-1\right)}$$
$$-\frac{(\epsilon-1)BC_{10}^{\mathrm{V}}\left(4x\epsilon-1\right)}{2\epsilon}+\frac{1}{2}BC_{11}^{\mathrm{V}}\left(4x\epsilon-6x+\frac{1}{\epsilon}-1\right)$$
$$+BC_{14}^{\mathrm{V}}\left(\frac{1}{2}-2x\epsilon\right)+BC_{15}^{\mathrm{V}}\left(1-4x\epsilon\right)$$
$$+BC_{16}^{\mathrm{V}}\left(4x\epsilon-2x+\frac{1}{2\epsilon}-1\right)$$
$$+\frac{BC_8^{\mathrm{V}}\left(-4x\epsilon^3+10x\epsilon^2-(8x+1)\epsilon+2x\left(2\epsilon^2-3\epsilon+1\right)\epsilon\log(x)+1\right)}{(\epsilon-1)\epsilon(4\epsilon-1)}$$
$$+\frac{(3\epsilon-2)BC_{12}^{\mathrm{V}}}{\epsilon^2\left(8\epsilon^2-6\epsilon+1\right)}\Big(44x\epsilon^4-6(9x+2)\epsilon^3+(25x+14)\epsilon^2$$
$$-(5x+6)\epsilon+2x\left(2\epsilon^2-3\epsilon+1\right)\epsilon^2\log(x)+1\Big)-\frac{(2\epsilon-1)BC_7^{\mathrm{V}}}{2(\epsilon-1)\epsilon(4\epsilon-1)}$$
$$\Big(-8x\epsilon^4-4x\epsilon^3+2(7x+1)\epsilon^2-3(2x+1)\epsilon+4x\left(2\epsilon^2-3\epsilon+1\right)\epsilon^2\log(x)+1\Big)$$
$$+\frac{(2\epsilon-1)BC_3^{\mathrm{V}}}{2(\epsilon-1)\epsilon^2(4\epsilon-1)}$$
$$\Big(-12x\epsilon^5-30x\epsilon^4+(86x+9)\epsilon^3-(58x+19)\epsilon^2$$
$$+2(5x+6)\epsilon+2x\left(6\epsilon^3-11\epsilon^2+6\epsilon-1\right)\epsilon^2\log(x)-2\Big)$$
$$+\frac{BC_1^{\mathrm{V}}}{4\epsilon^2(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)}$$
$$\Big(144x\epsilon^7+336x\epsilon^6-48(28x+3)\epsilon^5+13(106x+31)\epsilon^4-2(279x+203)\epsilon^3+14(7x+13)\epsilon^2$$
$$-2(3x+19)\epsilon-4x\left(36\epsilon^5-132\epsilon^4+169\epsilon^3-94\epsilon^2+23\epsilon-2\right)\epsilon^2\log(x)+3\Big)\Bigg)+\mathcal{O}(x^2)$$
$$\text{(B.15)}$$

$$
\begin{aligned}
M_{16}^{\mathrm{V}} = & \frac{BC_{13}^{\mathrm{V}}\left(-16x\epsilon^3+8x\epsilon-2x+2\epsilon^2+\epsilon-1\right)}{\epsilon(\epsilon+1)} \\
& + x^{-\epsilon}\left(\frac{(2\epsilon-1)BC_{13}^{\mathrm{V}}(x(6\epsilon-2)-1)}{\epsilon} + BC_{16}^{\mathrm{V}}(x(2-6\epsilon)+1)\right. \\
& \left. + \frac{BC_3^{\mathrm{V}}\left((\epsilon-1)\epsilon(x(6\epsilon-2)-1)\log(x)-2x\left(\epsilon^2+2\epsilon-1\right)\right)}{\epsilon}\right) \\
& x^{-2\epsilon}\left(x\left(4-\frac{2}{\epsilon}\right)BC_3^{\mathrm{V}}+2x(\epsilon-1)BC_1^{\mathrm{V}}\log(x)-2xBC_{10}^{\mathrm{V}}\right) \\
& + \mathcal{O}(x^2)
\end{aligned}
\tag{B.16}
$$

$$
\begin{aligned}
M_{17}^{\mathrm{V}} = & -\frac{(\epsilon-1)^2BC_1^{\mathrm{V}}}{2\epsilon^2(\epsilon+1)^2(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)}\Big(864x\epsilon^6-12(54x+5)\epsilon^5 \\
& + (12x-5)\epsilon^4+2(36x+59)\epsilon^3-6(2x-3)\epsilon^2-38\epsilon+7\Big)+\frac{(4-8\epsilon)BC_3^{\mathrm{V}}}{1-4\epsilon} \\
& + \frac{BC_5^{\mathrm{V}}}{2\epsilon^2(\epsilon+1)^2\left(6\epsilon^2-5\epsilon+1\right)}\Big(-216x\epsilon^5+(108x+55)\epsilon^4+8(3x+8)\epsilon^3 \\
& - 4(3x+7)\epsilon^2-28\epsilon+9\Big)+\frac{BC_6^{\mathrm{V}}}{\epsilon^2(\epsilon+1)^2\left(6\epsilon^2-5\epsilon+1\right)}\Big(216x\epsilon^5 \\
& - 4(27x+13)\epsilon^4-3(8x+23)\epsilon^3+(12x+11)\epsilon^2+21\epsilon-7\Big) \\
& - \frac{4(1-2\epsilon)^2BC_7^{\mathrm{V}}}{\epsilon(4\epsilon-1)}+\frac{(4-8\epsilon)BC_8^{\mathrm{V}}}{\epsilon^2-4\epsilon^3}+\left(2-\frac{2}{\epsilon}\right)BC_{10}^{\mathrm{V}}+\left(\frac{1}{\epsilon}-1\right)BC_{11}^{\mathrm{V}} \\
& - \frac{2(3\epsilon-2)BC_{12}^{\mathrm{V}}}{\epsilon^2(\epsilon+1)(2\epsilon+1)(4\epsilon-1)}\Big(216x\epsilon^5+(72-54x)\epsilon^4+(62-24x)\epsilon^3 \\
& + (6x-23)\epsilon^2-8\epsilon+5\Big)+2BC_{14}^{\mathrm{V}}+\left(\frac{2}{\epsilon}-4\right)BC_{16}^{\mathrm{V}}+BC_{17}^{\mathrm{V}}
\end{aligned}
$$

$$+x^{-\epsilon}\Bigg(-\frac{2(1-2\epsilon)^2BC_7^{\mathrm{V}}}{(\epsilon-1)\epsilon(4\epsilon-1)}\Big(6x\epsilon^2-\epsilon+1\Big)+\frac{4BC_8^{\mathrm{V}}}{\epsilon\left(4\epsilon^2-5\epsilon+1\right)}\Big($$

$$-6x\epsilon^2+\epsilon-1\Big)+2\,BC_{10}^{\mathrm{V}}\Big(6x\epsilon+\frac{1}{\epsilon}-1\Big)+BC_{11}^{\mathrm{V}}\Big(-12x\epsilon$$

$$-\frac{2}{\epsilon}+2\Big)-\frac{4\,BC_{12}^{\mathrm{V}}\left(6x\epsilon^2-\epsilon+1\right)}{(\epsilon-1)\epsilon^2(4\epsilon-1)}\Big(54\epsilon^3-69\epsilon^2+28\epsilon-4\Big)$$

$$+\frac{2BC_{14}^{\mathrm{V}}\left(6x\epsilon^2-\epsilon+1\right)}{\epsilon-1}-\frac{2(2\epsilon-1)BC_{16}^{\mathrm{V}}\left(6x\epsilon^2-\epsilon+1\right)}{(\epsilon-1)\epsilon}$$

$$-\frac{2(2\epsilon-1)BC_3^{\mathrm{V}}}{(\epsilon-1)\epsilon(\epsilon+1)(4\epsilon-1)}\Big(\epsilon\left(10x\epsilon^3+(30x+1)\epsilon^2+30x\epsilon\right.$$

$$\left.-10x-1\right)+2\left(4\epsilon^3-\epsilon^2-4\epsilon+1\right)\left(6x\epsilon^2-\epsilon+1\right)\log(x)\Big)$$

$$-\frac{4BC_6^{\mathrm{V}}}{(\epsilon-1)^2\epsilon^2\left(6\epsilon^3+\epsilon^2-4\epsilon+1\right)}\Big(-24x\epsilon^7+122x\epsilon^6$$

$$-(16x+17)\epsilon^5+(27-82x)\epsilon^4+(28x+5)\epsilon^3-(4x+25)\epsilon^2$$

$$+\left(6\epsilon^4-5\epsilon^3-5\epsilon^2+5\epsilon-1\right)\epsilon\left(6x\epsilon^2-\epsilon+1\right)\log(x)+12\epsilon$$

$$-2\Big)+\frac{BC_5^{\mathrm{V}}}{(\epsilon-1)^2\epsilon^2\left(6\epsilon^3+\epsilon^2-4\epsilon+1\right)}\Big(-96x\epsilon^7+302x\epsilon^6$$

$$-(10x+37)\epsilon^5+(68-154x)\epsilon^4+58x\epsilon^3-2(2x+31)\epsilon^2$$

$$+4\left(6\epsilon^4-5\epsilon^3-5\epsilon^2+5\epsilon-1\right)\epsilon\left(6x\epsilon^2-\epsilon+1\right)\log(x)$$

$$+37\epsilon-6\Big)+\frac{BC_1^{\mathrm{V}}}{\epsilon^2(\epsilon+1)(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)}\Big($$

$$-384x\epsilon^8+848x\epsilon^7-18(5x+4)\epsilon^6+(191-186x)\epsilon^5$$

$$+2(67x-48)\epsilon^4-2(19x+68)\epsilon^3+2(2x+81)\epsilon^2$$

$$+4\left(24\epsilon^5-26\epsilon^4-15\epsilon^3+25\epsilon^2-9\epsilon+1\right)\epsilon\left(6x\epsilon^2-\epsilon+1\right)\log(x)$$

$$-55\epsilon+6\Big)\Bigg)$$

$$
\begin{aligned}
&+x^{-2\epsilon}\Bigg(-\frac{\left(19\epsilon^2-16\epsilon+3\right)BC_5^{\mathrm{V}}}{2(1-2\epsilon)^2(\epsilon-1)\epsilon^2(3\epsilon-1)}\left(8x\epsilon^3-2(6x+1)\epsilon^2+3\epsilon\right. \qquad \text{(B.17)}\\
&\left.-1\right)+\frac{\left(16\epsilon^2-5\epsilon+1\right)BC_6^{\mathrm{V}}}{(1-2\epsilon)^2(\epsilon-1)\epsilon^2(3\epsilon-1)}\left(8x\epsilon^3-2(6x+1)\epsilon^2+3\epsilon-1\right)\\
&+\frac{BC_{11}^{\mathrm{V}}\left(-8x\epsilon^3+2(6x+1)\epsilon^2-3\epsilon+1\right)}{\epsilon-2\epsilon^2}-\frac{4BC_8^{\mathrm{V}}}{(\epsilon-1)\epsilon^2(2\epsilon-1)(4\epsilon-1)}\Big(\\
&-16x\epsilon^6+40x\epsilon^5-52x\epsilon^4+(34x+6)\epsilon^3-(4x+11)\epsilon^2\\
&+\left(2\epsilon^2-3\epsilon+1\right)\epsilon\left(4x\epsilon^3-10x\epsilon^2+2\epsilon-1\right)\log(x)+6\epsilon-1\Big)\\
&+\frac{2BC_7^{\mathrm{V}}}{(\epsilon-1)\epsilon(4\epsilon-1)}\Big(-32x\epsilon^6+64x\epsilon^5-44x\epsilon^4+2(7x+2)\epsilon^3\\
&+2(x-4)\epsilon^2+2\left(2\epsilon^2-3\epsilon+1\right)\epsilon\left(4x\epsilon^3-10x\epsilon^2+2\epsilon-1\right)\log(x)+5\epsilon\\
&-1\Big)-\frac{2(3\epsilon-2)BC_{12}^{\mathrm{V}}}{(1-2\epsilon)^2\epsilon^2(4\epsilon-1)}\Big(-32x\epsilon^6+64x\epsilon^5-44x\epsilon^4+2(7x+2)\epsilon^3\\
&+2(x-4)\epsilon^2+2\left(2\epsilon^2-3\epsilon+1\right)\epsilon\left(4x\epsilon^3-10x\epsilon^2+2\epsilon-1\right)\log(x)+5\epsilon-1\Big)\\
&-\frac{2BC_3^{\mathrm{V}}}{(\epsilon-1)\epsilon(4\epsilon-1)}\Big(\epsilon\left(-48x\epsilon^5+104x\epsilon^4-60x\epsilon^3+(2-14x)\epsilon^2+(28x-3)\epsilon\right.\\
&\left.-6x+1\right)+\left(6\epsilon^3-11\epsilon^2+6\epsilon-1\right)\left(4x\epsilon^3-10x\epsilon^2+2\epsilon-1\right)\log(x)\Big)\\
&+\frac{BC_1^{\mathrm{V}}}{2(1-2\epsilon)^2\epsilon^2\left(12\epsilon^2-7\epsilon+1\right)}\Big(-1152x\epsilon^9+3840x\epsilon^8-4896x\epsilon^7\\
&+8(485x+21)\epsilon^6-6(398x+111)\epsilon^5+(1020x+1043)\epsilon^4-12(19x+68)\epsilon^3\\
&+4(5x+83)\epsilon^2+2\left(6\epsilon^3-11\epsilon^2+6\epsilon-1\right)\epsilon\left(48x\epsilon^5-208x\epsilon^4\right.\\
&\left.+4(51x+8)\epsilon^3-2(16x+27)\epsilon^2+25\epsilon-3\right)\log(x)-66\epsilon+5\Big)\Bigg)\\
&+\mathcal{O}(x^2)
\end{aligned}
$$

B.2 Real-Virtual Contribution

$$
M_1^{\mathrm{R}}=r^{1-\epsilon}\,BC_1^{\mathrm{R}} \tag{B.18}
$$

$$
\begin{aligned}
M_2^{\mathrm{R}}=r^{-\epsilon}&\left(\frac{r(2-4\epsilon)BC_1^{\mathrm{R}}}{\epsilon-z\epsilon}\right)+\frac{r(1-3\epsilon)(2-3\epsilon)(1-2\epsilon)BC_3^{\mathrm{R}}}{(1-z)^2(1-\epsilon)\epsilon}-\frac{(2-3\epsilon)BC_3^{\mathrm{R}}}{(1-z)(1-\epsilon)}\\
&+r^{-2\epsilon}r\left(-\frac{2(1-\epsilon)^2BC_1^{\mathrm{R}}}{(1-z)(1-2\epsilon)\epsilon}-\frac{(2-3\epsilon)BC_3^{\mathrm{R}}}{(1-z)^2\epsilon}-\frac{(1-\epsilon)BC_2^{\mathrm{R}}}{1-2\epsilon}\right)+\mathcal{O}(r^2)
\end{aligned} \tag{B.19}
$$

$$M_3^{\mathrm{R}} = -\frac{(2-\epsilon(13-9(3-2\epsilon)\epsilon))\ BC_3^{\mathrm{R}}}{2(1-z)^2(1-\epsilon)\epsilon^2} + r^{-2\epsilon}\left(-\frac{(1-\epsilon)^2 BC_1^{\mathrm{R}}}{(1-z)\epsilon^2} \right.$$
$$\left. -\frac{(2-(7-6\epsilon)\epsilon)BC_3^{\mathrm{R}}}{2(1-z)^2\epsilon^2} - \frac{(1-\epsilon)BC_2^{\mathrm{R}}}{2\epsilon}\right) + r^{-\epsilon}\left(\frac{(\epsilon-1)(2\epsilon-1)BC_1^{\mathrm{R}}\log(r)}{(z-1)\epsilon}\right.$$
$$\left. + \frac{(2-3\epsilon)(1-2\epsilon)^2 BC_3^{\mathrm{R}}}{(1-z)^2(1-\epsilon)\epsilon^2} + \frac{(1-\epsilon)^2 BC_1^{\mathrm{R}}}{(1-z)\epsilon^2} - \frac{1}{2}\left(1-\frac{1}{\epsilon}\right)BC_2^{\mathrm{R}} + BC_{12}^{\mathrm{R}}\right) + \mathcal{O}(r) \tag{B.20}$$

$$M_4^{\mathrm{R}} = r^{-\epsilon}\,\frac{2rBC_1^{\mathrm{R}}}{z} + BC_4^{\mathrm{R}} + \frac{2r(2\epsilon-1)BC_4^{\mathrm{R}}}{z} + \mathcal{O}(r^2) \tag{B.21}$$

$$M_5^{\mathrm{R}} = \frac{2(2-(7-6\epsilon)\epsilon)BC_3^{\mathrm{R}}}{(1-z^2)\,(1-\epsilon)\epsilon} - \frac{(1-\epsilon)BC_1^{\mathrm{R}}}{(z+1)\epsilon} + \frac{(2-4\epsilon)BC_4^{\mathrm{R}}}{z\epsilon+\epsilon} + \frac{(1-2\epsilon)BC_{10}^{\mathrm{R}}}{z\epsilon+\epsilon}$$
$$+ \frac{BC_{11}^{\mathrm{R}}}{z\epsilon+\epsilon} - \frac{(1-z)BC_5^{\mathrm{R}}}{z+1} + r^{-\epsilon}\Big(-\frac{2\left(6\epsilon^2-7\epsilon+2\right)BC_3^{\mathrm{R}}}{(z^2-1)\,(\epsilon-1)\epsilon} + \frac{(1-\epsilon)BC_1^{\mathrm{R}}}{z\epsilon+\epsilon}$$
$$- \frac{(2-4\epsilon)BC_4^{\mathrm{R}}}{z\epsilon+\epsilon} - \frac{(1-2\epsilon)BC_{10}^{\mathrm{R}}}{z\epsilon+\epsilon} - \frac{BC_{11}^{\mathrm{R}}}{z\epsilon+\epsilon} + \frac{2BC_5^{\mathrm{R}}}{z+1} + \mathcal{O}(r)\Big)$$
$$+ r^{-2\epsilon+1}\Big(-\frac{2(\epsilon-1)BC_1^{\mathrm{R}}}{(z-1)z\epsilon} + \frac{\left(6\epsilon^2-7\epsilon+2\right)BC_3^{\mathrm{R}}}{(z-1)^2 z(\epsilon-1)\epsilon} - \frac{BC_2^{\mathrm{R}}}{z} + \mathcal{O}(r)\Big) \tag{B.22}$$

$$M_6^{\mathrm{R}} = BC_6^{\mathrm{R}} + \mathcal{O}(r) \tag{B.23}$$

$$M_7^{\mathrm{R}} = \left(\frac{1}{\epsilon}-2\right)BC_6^{\mathrm{R}} + r^{-\epsilon}\left((\epsilon-1)BC_1^{\mathrm{R}}\log(r) + \left(2-\frac{1}{\epsilon}\right)BC_6^{\mathrm{R}} + BC_7^{\mathrm{R}}\right) + \mathcal{O}(r) \tag{B.24}$$

$$M_8^{\mathrm{R}} = -\frac{(2\epsilon-1)\left(BC_4^{\mathrm{R}} - BC_6^{\mathrm{R}}\right)}{(z-1)\epsilon}$$
$$+ r^{-\epsilon}\left(\frac{(z-1)\epsilon BC_8^{\mathrm{R}} + (2\epsilon-1)BC_4^{\mathrm{R}} + (1-2\epsilon)BC_6^{\mathrm{R}}}{(z-1)\epsilon}\right) + \mathcal{O}(r) \tag{B.25}$$

$$
\begin{aligned}
M_9^{\mathrm{R}} = &\frac{(2\epsilon-1)BC_4^{\mathrm{R}}(z-4\epsilon+1)}{\left(1-z^2\right)\epsilon^2} - \frac{(1-2\epsilon)^2 BC_6^{\mathrm{R}}}{(z-1)\epsilon^2} + \frac{(1-2\epsilon)BC_{10}^{\mathrm{R}}}{z\epsilon+\epsilon} + \frac{(\epsilon-1)BC_1^{\mathrm{R}}}{(z+1)\epsilon} \\
&- \frac{\left(6\epsilon^2-7\epsilon+2\right)BC_3^{\mathrm{R}}((z+5)\epsilon-z-1)}{(z-1)^2(z+1)(\epsilon-1)\epsilon^2} + \frac{BC_{11}^{\mathrm{R}}}{z\epsilon+\epsilon} + \frac{(z-1)BC_5^{\mathrm{R}}}{z+1} \\
&+ r^{-\epsilon}\Bigg(\frac{2(\epsilon-1)BC_1^{\mathrm{R}}((2\epsilon-1)\epsilon\log(r)-\epsilon+1)}{(z-1)\epsilon^2} + \frac{2(3\epsilon-2)(1-2\epsilon)^2 BC_3^{\mathrm{R}}}{(z-1)^2(\epsilon-1)\epsilon^2} \\
&- \frac{2(1-2\epsilon)^2 BC_4^{\mathrm{R}}}{(z-1)\epsilon^2} + \frac{2(1-2\epsilon)^2 BC_6^{\mathrm{R}}}{(z-1)\epsilon^2} + \frac{(1-2\epsilon)BC_5^{\mathrm{R}}}{(z-1)\epsilon} + \frac{(1-2\epsilon)BC_7^{\mathrm{R}}}{\epsilon-z\epsilon} \\
&+ \frac{(1-2\epsilon)BC_{10}^{\mathrm{R}}}{(z-1)\epsilon} + \frac{1}{2}\left(\frac{1}{\epsilon}-1\right)BC_2^{\mathrm{R}} + \left(\frac{1}{\epsilon}-2\right)BC_8^{\mathrm{R}} + BC_{12}^{\mathrm{R}}\Bigg) \\
&+ r^{-2\epsilon}\Bigg(\frac{(\epsilon-1)BC_1^{\mathrm{R}}((z+3)\epsilon-2(z+1))}{\left(z^2-1\right)\epsilon^2} + \frac{(2\epsilon-1)BC_4^{\mathrm{R}}(z(4\epsilon-1)-1)}{\left(z^2-1\right)\epsilon^2} \\
&+ \frac{BC_5^{\mathrm{R}}\left(\left(z^2-4z-1\right)\epsilon+z+1\right)}{\epsilon-z^2\epsilon} + \frac{z(2-4\epsilon)BC_{10}^{\mathrm{R}}}{\epsilon-z^2\epsilon} - \frac{(1-2\epsilon)^2 BC_6^{\mathrm{R}}}{(z-1)\epsilon^2} \\
&+ \frac{(1-2\epsilon)BC_7^{\mathrm{R}}}{(z-1)\epsilon} - \frac{\left(6\epsilon^2-7\epsilon+2\right)BC_3^{\mathrm{R}}((3z-1)\epsilon-z-1)}{(z-1)^2(z+1)(\epsilon-1)\epsilon^2} \\
&- \frac{BC_{11}^{\mathrm{R}}}{z\epsilon+\epsilon} + \frac{(\epsilon-1)BC_2^{\mathrm{R}}}{2\epsilon} + \left(2-\frac{1}{\epsilon}\right)BC_8^{\mathrm{R}} + BC_9^{\mathrm{R}} - BC_{12}^{\mathrm{R}}\Bigg) + \mathcal{O}(r)
\end{aligned}
\tag{B.26}
$$

$$
\begin{aligned}
M_{10}^{\mathrm{R}} = {} & \frac{BC_{11}^{\mathrm{R}}}{(z^2-1)\,\epsilon} - \frac{3(1-2\epsilon)^2 BC_6^{\mathrm{R}}}{(z-1)^2\epsilon^2} + \frac{(1-2\epsilon)BC_7^{\mathrm{R}}}{(z-1)^2\epsilon} \\
& + \frac{(1-2\epsilon)BC_8^{\mathrm{R}}}{\epsilon - z\epsilon} - \frac{(z+3)(\epsilon-1)BC_1^{\mathrm{R}}}{(z-1)^2(z+1)\epsilon} + \frac{(1-\epsilon)BC_2^{\mathrm{R}}}{2\epsilon - 2z\epsilon} \\
& - \frac{\left(6\epsilon^2-7\epsilon+2\right)BC_3^{\mathrm{R}}((5z+9)\epsilon-3(z+1))}{(z-1)^3(z+1)(\epsilon-1)\epsilon^2} \\
& + \frac{(2\epsilon-1)BC_4^{\mathrm{R}}(4(z+2)\epsilon-3(z+1))}{(z-1)^2(z+1)\epsilon^2} + BC_5^{\mathrm{R}}\left(\frac{2\epsilon-1}{(z-1)^2\epsilon} + \frac{1}{z+1}\right) \\
& + \frac{(4\epsilon-2)BC_{10}^{\mathrm{R}}}{(z-1)^2(z+1)\epsilon} + \frac{BC_9^{\mathrm{R}}}{1-z} + \frac{BC_{12}^{\mathrm{R}}}{1-z} + BC_{16}^{\mathrm{R}} \\
& + r^{-\epsilon}\left(\frac{2(\epsilon-1)BC_1^{\mathrm{R}}(2\epsilon(2\epsilon-1)\log(r)+1)}{(z-1)^2\epsilon^2} + \frac{4(3\epsilon-2)(1-2\epsilon)^2 BC_3^{\mathrm{R}}}{(z-1)^3(\epsilon-1)\epsilon^2}\right. \\
& - \frac{4(1-2\epsilon)^2 BC_4^{\mathrm{R}}}{(z-1)^2\epsilon^2} + \frac{4(1-2\epsilon)^2 BC_6^{\mathrm{R}}}{(z-1)^2\epsilon^2} + \frac{(1-\epsilon)BC_2^{\mathrm{R}}}{(z-1)\epsilon} + \frac{(2-4\epsilon)BC_5^{\mathrm{R}}}{(z-1)^2\epsilon} \\
& \left. + \frac{(4\epsilon-2)BC_7^{\mathrm{R}}}{(z-1)^2\epsilon} + \frac{(2-4\epsilon)BC_8^{\mathrm{R}}}{(z-1)\epsilon} + \frac{(2-4\epsilon)BC_{10}^{\mathrm{R}}}{(z-1)^2\epsilon} + \frac{2BC_{12}^{\mathrm{R}}}{z-1}\right) \\
& + r^{-2\epsilon}\left(-\frac{BC_5^{\mathrm{R}}\left(\left(z^2-4z-1\right)\epsilon+z+1\right)}{(z-1)^2(z+1)\epsilon} + \frac{BC_{11}^{\mathrm{R}}}{\epsilon - z^2\epsilon} - \frac{(1-2\epsilon)^2 BC_6^{\mathrm{R}}}{(z-1)^2\epsilon^2}\right. \\
& + \frac{(1-2\epsilon)BC_7^{\mathrm{R}}}{(z-1)^2\epsilon} + \frac{(1-2\epsilon)BC_8^{\mathrm{R}}}{\epsilon - z\epsilon} + \frac{(\epsilon-1)BC_1^{\mathrm{R}}((z+3)\epsilon-2(z+1))}{(z-1)^2(z+1)\epsilon^2} \\
& + \frac{(1-\epsilon)BC_2^{\mathrm{R}}}{2\epsilon-2z\epsilon} - \frac{\left(6\epsilon^2-7\epsilon+2\right)BC_3^{\mathrm{R}}((3z-1)\epsilon-z-1)}{(z-1)^3(z+1)(\epsilon-1)\epsilon^2} \\
& \left. + \frac{(2\epsilon-1)BC_4^{\mathrm{R}}(z(4\epsilon-1)-1)}{(z-1)^2(z+1)\epsilon^2} + \frac{2z(2\epsilon-1)BC_{10}^{\mathrm{R}}}{(z-1)^2(z+1)\epsilon} + \frac{BC_9^{\mathrm{R}}}{z-1} + \frac{BC_{12}^{\mathrm{R}}}{1-z}\right) + \mathcal{O}(r)
\end{aligned}
\tag{B.27}
$$

$$
\begin{aligned}
M_{11}^{\mathrm{R}} = {} & \frac{(1-\epsilon)BC_1^{\mathrm{R}}}{z\epsilon+\epsilon} + \frac{\left(6\epsilon^2-7\epsilon+2\right)BC_3^{\mathrm{R}}}{(z+1)(\epsilon-1)\epsilon} - \frac{(z-1)(2\epsilon-1)BC_4^{\mathrm{R}}}{(z+1)\epsilon} \\
& + \frac{(2\epsilon-1)BC_{10}^{\mathrm{R}}}{z\epsilon+\epsilon} - \frac{BC_{11}^{\mathrm{R}}}{z\epsilon+\epsilon} + \left(\frac{2}{z+1}-1\right)BC_5^{\mathrm{R}} \\
& + r^{-\epsilon}\left(\frac{(\epsilon-1)BC_1^{\mathrm{R}}}{(z+1)\epsilon} + \frac{\left(-6\epsilon^2+7\epsilon-2\right)BC_3^{\mathrm{R}}}{(z+1)(\epsilon-1)\epsilon} + \frac{(z-1)(2\epsilon-1)BC_4^{\mathrm{R}}}{(z+1)\epsilon}\right. \\
& \left. + \frac{BC_{10}^{\mathrm{R}}(z\epsilon-\epsilon+1)}{z\epsilon+\epsilon} + \frac{BC_{11}^{\mathrm{R}}}{z\epsilon+\epsilon} + \frac{(z-1)BC_5^{\mathrm{R}}}{z+1}\right) \\
& + r^{-2\epsilon+1}\left(-\frac{2(\epsilon-1)BC_1^{\mathrm{R}}}{z\epsilon} + \frac{\left(6\epsilon^2-7\epsilon+2\right)BC_3^{\mathrm{R}}}{(z-1)z(\epsilon-1)\epsilon} + \left(\frac{1}{z}-1\right)BC_2^{\mathrm{R}}\right) \\
& + \mathcal{O}(r)
\end{aligned}
\tag{B.28}
$$

$$
\begin{aligned}
M_{12}^{\mathrm{R}} = {} & \frac{(\epsilon-1)BC_1^{\mathrm{R}}}{(z+1)\epsilon} + \frac{\left(-6\epsilon^2+7\epsilon-2\right)BC_3^{\mathrm{R}}}{(z+1)(\epsilon-1)\epsilon} + \frac{(z-1)(2\epsilon-1)BC_4^{\mathrm{R}}}{(z+1)\epsilon} \\
& + \frac{(1-2\epsilon)BC_{10}^{\mathrm{R}}}{z\epsilon+\epsilon} + \frac{BC_{11}^{\mathrm{R}}}{z\epsilon+\epsilon} + \frac{(z-1)BC_5^{\mathrm{R}}}{z+1} \\
& + r^{-\epsilon}\left(\frac{(1-\epsilon)BC_1^{\mathrm{R}}}{z\epsilon+\epsilon} + \frac{\left(6\epsilon^2-7\epsilon+2\right)BC_3^{\mathrm{R}}}{(z+1)(\epsilon-1)\epsilon} + \frac{BC_4^{\mathrm{R}}(-2(z-1)\epsilon+z-1)}{(z+1)\epsilon}\right. \\
& \left. - \frac{(1-2\epsilon)BC_{10}^{\mathrm{R}}}{z\epsilon+\epsilon} + \frac{BC_{11}^{\mathrm{R}}(z\epsilon+\epsilon-1)}{(z+1)\epsilon} + \frac{(1-z)BC_5^{\mathrm{R}}}{z+1}\right) \\
& + r^{-2\epsilon+2}\left(\frac{2(\epsilon-1)BC_1^{\mathrm{R}}(z(\epsilon-2)+1)}{(z-1)z^2(\epsilon-2)\epsilon} + \frac{BC_2^{\mathrm{R}}(z(\epsilon-2)+1)}{z^2(\epsilon-2)}\right. \\
& \left. - \frac{\left(6\epsilon^2-7\epsilon+2\right)BC_3^{\mathrm{R}}(z(\epsilon-2)+1)}{(z-1)^2z^2(\epsilon-2)(\epsilon-1)\epsilon}\right) + \mathcal{O}(r)
\end{aligned}
\tag{B.29}
$$

$$
\begin{aligned}
M_{13}^{\mathrm{R}} = {} & BC_3^{\mathrm{R}}\left(\frac{r\left(6\epsilon^2-7\epsilon+2\right)}{(z-1)\epsilon}+1\right)+r^{-\epsilon+1}\left(\frac{2(\epsilon-1)BC_1^{\mathrm{R}}}{\epsilon}\right)\\
& + r^{-2\epsilon+1}\Bigg(-\frac{r(z-1)(\epsilon-1)BC_2^{\mathrm{R}}}{2\epsilon-1}+\frac{r(3\epsilon-2)BC_3^{\mathrm{R}}}{(z-1)\epsilon}\\
& + r\left(\frac{2}{\epsilon}+\frac{1}{1-2\epsilon}-1\right)BC_1^{\mathrm{R}}\Bigg)+\mathcal{O}(r^2)
\end{aligned}
\tag{B.30}
$$

$$
\begin{aligned}
M_{14}^{\mathrm{R}} = {} & \Bigg(-\frac{4(\epsilon-1)BC_1^{\mathrm{R}}}{(z-1)^4 z\epsilon^2(\epsilon+1)^2}\Big(\left(z^3-z^2-2z-2\right)\epsilon^3+\left(2z^3-2z^2+z+3\right)\epsilon^2\\
& + \left(z^3-z^2-2z+4\right)\epsilon+2(z-1)\epsilon^4-z+1\Big)\\
& + \frac{2\left(6\epsilon^2-7\epsilon+2\right)BC_3^{\mathrm{R}}}{(1-z)^5\epsilon^2\left(z-z\epsilon^2\right)}\Big(z^3(3-7\epsilon)(\epsilon+1)+8z^2\left(2\epsilon^2+\epsilon-1\right)\\
& + z(\epsilon-4)\epsilon(2\epsilon-3)+z-2\epsilon^2(\epsilon+1)\Big)\\
& + \frac{4(2\epsilon-1)BC_4^{\mathrm{R}}}{(z-1)^4 z\epsilon^2(\epsilon+1)}\left(-z^3+3z^2+\left(z^3-z^2-5z+1\right)\epsilon+2(z-1)^2 z\epsilon^2-z+1\right)\\
& + \frac{4BC_5^{\mathrm{R}}}{(z-1)^4 z\epsilon(\epsilon+1)}\left(-z^3+\left(z^2-6z+1\right)\epsilon+2z^2+(z-1)^2(z+1)\epsilon^2+z\right)\\
& + \frac{4(2\epsilon-1)BC_{10}^{\mathrm{R}}}{(z-1)^4 z\epsilon(\epsilon+1)}\left(z^3-z^2+(z-1)^2(z+1)\epsilon-3z+1\right)\\
& - \frac{2(\epsilon-1)(2\epsilon+1)BC_2^{\mathrm{R}}((z-1)\epsilon-z-1)}{(z-1)^3 z\epsilon(\epsilon+1)}-\frac{4BC_{11}^{\mathrm{R}}}{(z-1)^2 z\epsilon}+\frac{4BC_{12}^{\mathrm{R}}\left((z-1)^2-\frac{2}{\epsilon+1}\right)}{(1-z)^3}\\
& + BC_{13}^{\mathrm{R}}\left(2-\frac{2(6\epsilon+1)}{(z-1)^2\epsilon}\right)+\frac{(4\epsilon+2)BC_{14}^{\mathrm{R}}}{(z-1)\epsilon}-\frac{2(z-3)(z+1)BC_{15}^{\mathrm{R}}}{(z-1)^3}\Bigg)\\
& + r^{-\epsilon}\Bigg(\frac{4(\epsilon-1)BC_1^{\mathrm{R}}(\epsilon(2\epsilon-1)\log(r)+1)}{(z-1)^2\epsilon^2}+\frac{8(3\epsilon-2)(1-2\epsilon)^2 BC_3^{\mathrm{R}}}{(z-1)^3(\epsilon-1)\epsilon^2}\\
& - \frac{4(1-2\epsilon)^2 BC_4^{\mathrm{R}}}{(z-1)^2\epsilon^2}-\frac{2(\epsilon-1)BC_2^{\mathrm{R}}}{(z-1)\epsilon}+\frac{(4-8\epsilon)BC_5^{\mathrm{R}}}{(z-1)^2\epsilon}+\frac{(4-8\epsilon)BC_{10}^{\mathrm{R}}}{(z-1)^2\epsilon}+\frac{4BC_{12}^{\mathrm{R}}}{z-1}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+r^{-2\epsilon}\Bigg(-\frac{2\left(6\epsilon^2-7\epsilon+2\right)BC_3^{\mathrm{R}}\left(-z\left(z^2+3\right)+2(z-1)\epsilon^3+(z-2)(z+1)^2\epsilon^2+16z\epsilon\right)}{(z-1)^5z(\epsilon-1)\epsilon^2(\epsilon+1)}\\
&\qquad-\frac{4(2\epsilon-1)BC_4^{\mathrm{R}}\left(\left(z^2-6z+1\right)\epsilon+z^2+1\right)}{(z-1)^4z\epsilon^2(\epsilon+1)}\\
&\qquad-\frac{4BC_{10}^{\mathrm{R}}\left(\left(z^2-6z+1\right)\epsilon-z^2+2(z-1)^2\epsilon^2+4z-1\right)}{(z-1)^4z\epsilon(\epsilon+1)}+\frac{2\left(z^2-2z-3\right)BC_{15}^{\mathrm{R}}}{(z-1)^3}\\
&\qquad+\frac{4(\epsilon-1)BC_1^{\mathrm{R}}}{(z-1)^4z\epsilon^2(\epsilon+1)^2}\Big(\left(z^3+3\right)\epsilon^2-z^3+2z^2+\left(z^3-z^2-2z-2\right)\epsilon^3\\
&\qquad\qquad-\left(z^3-3z^2+4z-4\right)\epsilon+2(z-1)\epsilon^4-2z+1\Big)\\
&\qquad+\frac{BC_2^{\mathrm{R}}\left(-2z^3+4z^2+2\left(z^3-2z^2-2z-1\right)\epsilon^2+4(z-1)\epsilon^3+4\epsilon+2\right)}{(z-1)^3z\epsilon(\epsilon+1)}\\
&\qquad+\frac{4BC_5^{\mathrm{R}}\left(\left(z^3-3z^2+7z-1\right)\epsilon+(z-1)^3\epsilon^2-2z\right)}{(z-1)^4z\epsilon(\epsilon+1)}+\frac{4BC_{11}^{\mathrm{R}}}{(z-1)^2z\epsilon}\\
&\qquad-\frac{8BC_{12}^{\mathrm{R}}}{(z-1)^3(\epsilon+1)}+BC_{13}^{\mathrm{R}}\left(\frac{2}{(z-1)^2\epsilon}+\frac{12}{(z-1)^2}-1\right)+\frac{(4\epsilon+2)BC_{14}^{\mathrm{R}}}{\epsilon-z\epsilon}\Bigg)\\
&+\mathcal{O}(r) \qquad\qquad (\mathrm{B.31})
\end{aligned}
$$

$$
\begin{aligned}
M_{15}^{\mathrm{R}} = \Bigg(& - \frac{2BC_1^{\mathrm{R}}(\epsilon-1)}{(z-1)^5\epsilon^2(\epsilon+1)^2(2z\epsilon+z)}\Big(2\left(z^3-z^2-6z+8\right)\epsilon \\
& - \left(z^4-12z^3+14z^2+8z+21\right)\epsilon^4 - 2\left(z^4-13z^3+15z^2-6z-9\right)\epsilon^3 \\
& - \left(z^4-16z^3+18z^2+10z-37\right)\epsilon^2 + 16(z-1)\epsilon^5 - 2z \\
& + 2\Big)\Bigg) - \frac{2BC_2^{\mathrm{R}}(\epsilon-1)(4\epsilon+1)((z-1)\epsilon-z-1)}{(z-1)^4z\epsilon(\epsilon+1)} \\
& + BC_3^{\mathrm{R}}\Bigg(\frac{\left(6\epsilon^2-7\epsilon+2\right)}{(z-1)^6z(\epsilon-1)\epsilon^2(\epsilon+1)(2\epsilon+1)}\Big(2z\left(3z^2-8z+1\right) \\
& + \left(-48z^3+102z^2+68z-2\right)\epsilon^2 - \left(2z^4+51z^3-125z^2+85z+19\right)\epsilon^3 \\
& + \left(2z^4+9z^3-39z^2+27z+1\right)\epsilon + 16(z-1)\epsilon^4\Big)\Bigg) \\
& + BC_4^{\mathrm{R}}\Bigg(\frac{2(1-2\epsilon)}{(1-z)^5z\epsilon^2(\epsilon+1)(2\epsilon+1)}\Big(z^4(-\epsilon)(\epsilon+1) + 2z^3(\epsilon+1)\left(8\epsilon^2-1\right) \\
& + 2z^2(\epsilon+1)\left(-16\epsilon^2+5\epsilon+3\right) - 2z\epsilon(7-8(\epsilon-2)\epsilon) - 2z + \epsilon(7\epsilon+9) + 2\Big)\Bigg) \\
& + BC_5^{\mathrm{R}}\Bigg(- \frac{1}{(1-z)^5z\epsilon(\epsilon+1)(2\epsilon+1)}\Bigg(2\left((z-1)^2\left(z^3-3z^2+11z+7\right)\epsilon^3\right. \\
& \qquad + (z-3)(z(z((z-2)z+6)+14)-3)\epsilon^2 + (2(1-4z)(z-2)z+2)\epsilon \\
& \qquad \left. + 2z(1-(z-2)z)\right)\Bigg)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+BC_{10}^{\mathrm{R}}\Bigg(-\frac{2(1-2\epsilon)}{(1-z)^5 z\epsilon(\epsilon+1)(2\epsilon+1)}\Bigg(\left(\left(z^2-10z-7\right)(z-1)^2\epsilon^2\right.\\
&\left.+2z\left(-z^2+z+3\right)+(z+1)(z((z-15)z+31)-9)\epsilon-2\right)\Bigg)\Bigg)\\
&+BC_{11}^{\mathrm{R}}\left(\frac{2\left(\left(z^2-2z-7\right)\epsilon-2\right)}{(z-1)^3 z\epsilon(2\epsilon+1)}\right)+BC_{12}^{\mathrm{R}}\Bigg(\\
&-\frac{4(4\epsilon+1)\left(z^2+(z-1)^2\epsilon-2z-1\right)}{(z-1)^4(\epsilon+1)(2\epsilon+1)}\Bigg)\\
&+BC_{13}^{\mathrm{R}}\left(\frac{2(4\epsilon+1)\left(\left(z^2-2z-5\right)\epsilon-1\right)}{(z-1)^3\epsilon(2\epsilon+1)}\right)\\
&+BC_{14}^{\mathrm{R}}\left(\frac{8\epsilon+2}{(z-1)^2\epsilon}\right)+BC_{15}^{\mathrm{R}}\left(-\frac{2(z-3)(z+1)(4\epsilon+1)}{(z-1)^4(2\epsilon+1)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+r^{-\epsilon}\Bigg(\frac{4BC_1^{\mathrm{R}}\left(\left(4\epsilon^5-5\epsilon^3+\epsilon\right)\log(r)+4\epsilon^3-2\epsilon^2-\epsilon-1\right)}{(z-1)^3\epsilon^2(\epsilon+1)^2}\\
&\quad+\frac{8\left(6\epsilon^2-\epsilon-2\right)(1-2\epsilon)^2 BC_3^{\mathrm{R}}}{(z-1)^4\epsilon^2\left(\epsilon^2-1\right)}-\frac{4(2\epsilon+1)(1-2\epsilon)^2 BC_4^{\mathrm{R}}}{(z-1)^3\epsilon^2(\epsilon+1)}\\
&\quad+\frac{\left(-4\epsilon^2+2\epsilon+2\right)BC_2^{\mathrm{R}}}{(z-1)^2\epsilon(\epsilon+1)}+\frac{\left(4-16\epsilon^2\right)BC_5^{\mathrm{R}}}{(z-1)^3\epsilon(\epsilon+1)}\\
&\quad+\frac{\left(4-16\epsilon^2\right)BC_{10}^{\mathrm{R}}}{(z-1)^3\epsilon(\epsilon+1)}+\frac{(8\epsilon+4)BC_{12}^{\mathrm{R}}}{(z-1)^2(\epsilon+1)}\Bigg)
\end{aligned}
$$

$$+r^{-2\epsilon-1}\Bigg(+ BC_1^{\mathrm{R}}\Bigg($$

$$-\frac{2(\epsilon-1)\left(\left(z^2-3z-2\right)\epsilon^3+\left(2z^2-z+3\right)\epsilon^2+\left(z^2-3z+4\right)\epsilon+2(z-1)\epsilon^4-z+1\right)}{(z-1)^3 z\epsilon(\epsilon+1)^2(2\epsilon+1)}\Bigg)$$

$$+ BC_2^{\mathrm{R}}\left(-\frac{(\epsilon-1)((z-1)\epsilon-z-1)}{(z-1)^2 z(\epsilon+1)}\right)$$

$$+ BC_3^{\mathrm{R}}\left(\frac{2\left(6\epsilon^2-7\epsilon+2\right)\left(\left(z^2-2z-1\right)\epsilon^2+(z-1)\epsilon^3+8z\epsilon-z(z+1)\right)}{(z-1)^4 z(\epsilon-1)\epsilon(\epsilon+1)(2\epsilon+1)}\right)$$

$$+ BC_4^{\mathrm{R}}\left(\frac{2(2\epsilon-1)\left(\left(z^2-6z+1\right)\epsilon+z^2+1\right)}{(z-1)^3 z\epsilon(\epsilon+1)(2\epsilon+1)}\right)$$

$$+ BC_5^{\mathrm{R}}\left(\frac{2\left(-\left(z^3-3z^2+7z-1\right)\epsilon+(z-1)^3\left(-\epsilon^2\right)+2z\right)}{(z-1)^3 z(\epsilon+1)(2\epsilon+1)}\right)$$

$$+ BC_{10}^{\mathrm{R}}\left(\frac{2\left(\left(z^2-6z+1\right)\epsilon-z^2+2(z-1)^2\epsilon^2+4z-1\right)}{(z-1)^3 z(\epsilon+1)(2\epsilon+1)}\right)-\frac{2BC_{11}^{\mathrm{R}}}{(z-1)z(2\epsilon+1)}$$

$$+\frac{4BC_{12}^{\mathrm{R}}\epsilon}{(z-1)^2\left(2\epsilon^2+3\epsilon+1\right)}+\frac{BC_{13}^{\mathrm{R}}(6\epsilon+1)}{-2z\epsilon-z+2\epsilon+1}+BC_{14}^{\mathrm{R}}+\frac{4BC_{15}^{\mathrm{R}}\,\epsilon}{(z-1)^2(2\epsilon+1)}\Bigg)$$

$$M_{16}^{\mathrm{R}} = BC_1^{\mathrm{R}} \Bigg(-\frac{2(\epsilon-1)}{(z-1)^3 z\epsilon^2(\epsilon+1)^2} \Big(\left(z^3-z^2-2z-2\right)\epsilon^3$$

$$+\left(2z^3-2z^2+z+3\right)\epsilon^2+\left(z^3-z^2-2z+4\right)\epsilon+2(z-1)\epsilon^4$$

$$-z+1\Big)+\frac{BC_2^{\mathrm{R}}\,(\epsilon-1)(2\epsilon+1)(-z\epsilon+z+\epsilon+1)}{(z-1)^2 z\epsilon(\epsilon+1)}$$

$$+\frac{BC_3^{\mathrm{R}}\,\left(6\epsilon^2-7\epsilon+2\right)}{(z-1)^4 z\epsilon^2\left(\epsilon^2-1\right)}\Big(-4z\left(z^2-2z-3\right)\epsilon+z\left(3z^2-8z+1\right)$$

$$-\left(7z^3-16z^2+11z+2\right)\epsilon^2+2(z-1)\epsilon^3\Big)$$

$$-\frac{2BC_4^{\mathrm{R}}\,(1-2\epsilon)\left(\frac{2-4\epsilon}{(z-1)^3(\epsilon+1)}+\frac{2\epsilon}{z-1}-\frac{1}{z}\right)}{\epsilon^2}+BC_5^{\mathrm{R}}\Big(\frac{4-8\epsilon}{(z-1)^3\epsilon(\epsilon+1)}$$

$$+\frac{2-4\epsilon}{\epsilon-z\epsilon}-\frac{2}{z}\Big)+\frac{2BC_{10}^{\mathrm{R}}\,(2\epsilon-1)\left(z^3-z^2+(z-1)^2(z+1)\epsilon-3z+1\right)}{(z-1)^3 z\epsilon(\epsilon+1)}$$

$$+\frac{2\,BC_{11}^{\mathrm{R}}}{z\epsilon-z^2\epsilon}+BC_{12}^{\mathrm{R}}\Big(\frac{4}{(z-1)^2(\epsilon+1)}-2\Big)+BC_{13}^{\mathrm{R}}\Big(\frac{(z+5)\epsilon+1}{\epsilon-z\epsilon}$$

$$+z\Big)+BC_{14}^{\mathrm{R}}\Big(\frac{1}{\epsilon}+2\Big)+BC_{15}^{\mathrm{R}}\Big(\frac{4}{(z-1)^2}-1\Big)\Bigg)$$

$$+r^{-\epsilon}\Bigg(\frac{2(\epsilon-1)BC_1^{\mathrm{R}}(\epsilon(2\epsilon-1)\log(r)+1)}{(z-1)\epsilon^2}+\frac{4(3\epsilon-2)(1-2\epsilon)^2 BC_3^{\mathrm{R}}}{(z-1)^2(\epsilon-1)\epsilon^2}$$

$$-\frac{2(1-2\epsilon)^2 BC_4^{\mathrm{R}}}{(z-1)\epsilon^2}+\frac{(2-4\epsilon)BC_5^{\mathrm{R}}}{(z-1)\epsilon}+\frac{(2-4\epsilon)BC_{10}^{\mathrm{R}}}{(z-1)\epsilon}+\Big(\frac{1}{\epsilon}-1\Big)BC_2^{\mathrm{R}}+2BC_{12}^{\mathrm{R}}\Bigg)$$

$$
\begin{aligned}
&+r^{-2\epsilon}\Bigg(\frac{2BC_1^{\mathrm{R}}\,(\epsilon-1)}{(z-1)^3 z\epsilon^2(\epsilon+1)^2}\left(\left(z^3+3\right)\epsilon^2-z^3+2z^2\right.\\
&\quad+\left(z^3-z^2-2z-2\right)\epsilon^3-\left(z^3-3z^2+4z-4\right)\epsilon+2(z-1)\epsilon^4-2z+1\Big)\\
&\quad+BC_2^{\mathrm{R}}\left(\frac{-z^3+2z^2+\left(z^3-2z^2-2z-1\right)\epsilon^2+2(z-1)\epsilon^3+2\epsilon+1}{(z-1)^2 z\epsilon(\epsilon+1)}\right)\\
&\quad-\frac{BC_3^{\mathrm{R}}\left(6\epsilon^2-7\epsilon+2\right)}{(z-1)^4 z(\epsilon-1)\epsilon^2(\epsilon+1)}\left(-z\left(z^2+3\right)+2(z-1)\epsilon^3\right.\\
&\quad\left.+(z-2)(z+1)^2\epsilon^2+16z\epsilon\right)-\frac{2BC_4^{\mathrm{R}}\,(2\epsilon-1)\left(\left(z^2-6z+1\right)\epsilon+z^2+1\right)}{(z-1)^3 z\epsilon^2(\epsilon+1)}\\
&\quad+BC_5^{\mathrm{R}}\left(-\frac{4}{(z-1)^3\epsilon(\epsilon+1)}+\frac{8}{(z-1)^3(\epsilon+1)}+\frac{2}{z}\right)\\
&\quad-\frac{2\,BC_{10}^{\mathrm{R}}\left(\left(z^2-6z+1\right)\epsilon-z^2+2(z-1)^2\epsilon^2+4z-1\right)}{(z-1)^3 z\epsilon(\epsilon+1)}\\
&\quad-\frac{2\,BC_{11}^{\mathrm{R}}}{z\epsilon-z^2\epsilon}-\frac{4\,BC_{12}^{\mathrm{R}}}{(z-1)^2(\epsilon+1)}+BC_{13}^{\mathrm{R}}\left(\frac{\left(-z^2+2z+5\right)\epsilon+1}{(z-1)\epsilon}\right)\\
&\quad+BC_{14}^{\mathrm{R}}\left(-\frac{1}{\epsilon}-2\right)+BC_{15}^{\mathrm{R}}\left(2-\frac{4}{(z-1)^2}\right)\Bigg)
\end{aligned}
$$

Appendix C

Boundary Conditions

C.1 Boundary Conditions in Double-Virtual Contribution

$$BC_1^{\mathrm{V}} = -\frac{1}{\epsilon^2} - \frac{2}{\epsilon} + \left(-3 - \frac{\pi^2}{6}\right) + \left(-4 - \frac{\pi^2}{3} + \frac{2}{3}\zeta_3\right)\epsilon + \mathcal{O}(\epsilon^2) \tag{C.1}$$

$$BC_2^{\mathrm{V}} = -\frac{1}{\epsilon^2} - \frac{3}{\epsilon} - 7 + \Big(- 15 + \frac{8}{3}\zeta_3\Big)\epsilon + \mathcal{O}(\epsilon^2) \tag{C.2}$$

$$BC_3^{\mathrm{V}} = BC_2^{\mathrm{V}} \tag{C.3}$$

$$BC_4^{\mathrm{V}} = \Bigg(- \frac{1}{\epsilon^2} - \frac{4}{\epsilon} + \left(-72 + \pi^2\right) + \frac{2}{3}\Big(- 48 + \pi^2 + 7\zeta_3\Big)\epsilon \\ + \mathcal{O}(\epsilon^2)\Bigg) \tag{C.4}$$

$$BC_5^{\mathrm{V}} = \Big(- 6\zeta_3\,\epsilon + \mathcal{O}(\epsilon^2)\Big) \tag{C.5}$$

$$BC_6^{\mathrm{V}} = \Big(- \frac{\pi^2}{6} - 8\zeta_3\,\epsilon + \mathcal{O}(\epsilon^2)\Big) \tag{C.6}$$

$$BC_7^{\mathrm{V}} = \Bigg(- \frac{1}{2\epsilon^2} - \frac{5}{2\epsilon} + \Big(- \frac{23}{2} - \frac{\pi^2}{12} - 4\zeta_3\Big)$$

$$+\Bigg(-\frac{97}{2}-\frac{7\pi^2}{12}+\frac{11\pi^4}{90}+\frac{10\zeta_3}{3}\Bigg)\epsilon+\mathcal{O}(\epsilon^2)\Bigg) \tag{C.7}$$

$$BC_8^{\mathrm{V}}=\Bigg(\frac{9}{2}\zeta_3\epsilon+\mathcal{O}(\epsilon^2)\Bigg) \tag{C.8}$$

$$BC_9^{\mathrm{V}}=\Bigg(6\zeta_3+\Bigg(\frac{\pi^4}{10}+12\zeta_3\Bigg)\epsilon+\mathcal{O}(\epsilon^2)\Bigg) \tag{C.9}$$

$$BC_{10}^{\mathrm{V}}=\Bigg(-3\zeta_3+\Bigg(-\frac{\pi^4}{72}-3\zeta_3\Bigg)\epsilon+\mathcal{O}(\epsilon^2)\Bigg) \tag{C.10}$$

$$\begin{aligned}BC_{11}^{\mathrm{V}}=\Bigg(&\frac{3}{2\epsilon^2}+\frac{3}{2\epsilon}+\frac{1}{4}\left(-30+\pi^2+16\zeta_3\right)\\&+\frac{1}{180}\Bigg(-9450-255\pi^2-8\pi^4\Bigg)\epsilon+\mathcal{O}(\epsilon^2)\Bigg)\end{aligned} \tag{C.11}$$

$$\begin{aligned}BC_{12}^{\mathrm{V}}=\Bigg(&-\frac{1}{8\epsilon^2}-\frac{9}{16\epsilon}+\Bigg(-\frac{63}{32}-\frac{\pi^2}{48}\Bigg)\\&+\Bigg(-\frac{405}{64}-\frac{3\pi^2}{32}+\frac{13\zeta_3}{12}\Bigg)\epsilon+\mathcal{O}(\epsilon^2)\Bigg)\end{aligned} \tag{C.12}$$

$$\begin{aligned}BC_{13}^{\mathrm{V}}=\Bigg(&-\frac{1}{\epsilon^2}-\frac{4}{\epsilon}+\left(-72+\pi^2\right)+\frac{2}{3}\Bigg(-48+\pi^2+7\zeta_3\Bigg)\epsilon\\&+\mathcal{O}(\epsilon^2)\Bigg)\end{aligned} \tag{C.13}$$

$$BC_{14}^{\mathrm{V}}=\Bigg(-\frac{\pi^4}{10}+\Bigg(2\pi^2\zeta_3-7\zeta_5\Bigg)\epsilon+\mathcal{O}(\epsilon^2)\Bigg) \tag{C.14}$$

$$BC_{15}^{\mathrm{V}} = \left(\frac{\pi^4}{36} + \left(\frac{1}{2}\pi^2\zeta_3 - 9\zeta_5\right)\epsilon + \mathcal{O}(\epsilon^2)\right) \tag{C.15}$$

$$BC_{16}^{\mathrm{V}} = \left(-3\zeta_3 + \left(-\frac{\pi^4}{72} - 6\zeta_3\right)\epsilon + \mathcal{O}(\epsilon^2)\right) \tag{C.16}$$

C.2 Boundary Conditions in Real-Virtual Contribution

In this section we will present the boundary conditions as an expansion in ϵ. For convenience we use the following notation for Harmonic Polylogarithms:

$$H\left(\vec{a}; z\right) = H_{\vec{a}} \tag{C.17}$$

$$\begin{aligned} BC_1^{\mathrm{R}} = \Bigg(&\frac{1}{\epsilon} + 1 + \left(\frac{\pi^2}{12} + 1\right)\epsilon + \frac{1}{12}\left(-4\zeta_3 + 12 + \pi^2\right)\epsilon^2 \\ &+ \frac{1}{480}\left(-160(\zeta_3 - 3) + 40\pi^2 + 3\pi^4\right)\epsilon^3 + \mathcal{O}(\epsilon^4)\Bigg) \end{aligned} \tag{C.18}$$

$$\begin{aligned} BC_2^{\mathrm{R}} = \Bigg(&\frac{\log^2(1-z) - 2\log(1-z) + 2}{1-z} + \frac{2\epsilon\left(12 - \pi^2 + \left(-6 + \pi^2\right)\log(1-z) - 6\zeta_3\right)}{3(1-z)} \\ &\frac{\epsilon^2}{180(1-z)}\Big(+ 360(4\zeta_3 - 13) + 4\pi^4 + 210\pi^2 + 30\log(1-z)\left(-48\zeta_3 + 60 + \pi^2\right) \\ &- 15\log^2(1-z)\left(12 + \pi^2\right) + 60\log^3(1-z) - 15\log^4(1-z)\Big) + \mathcal{O}(\epsilon^3)\Bigg) \end{aligned} \tag{C.19}$$

$$\begin{aligned} BC_3^{\mathrm{R}} = \Bigg(&-\frac{1-z}{2\epsilon} + \frac{1}{4}(1-z)\left(2\log(1-z) - 7\right) \\ &+ \frac{\epsilon}{8}(1-z)\left(\pi^2 - 45 + 14\log(1-z) - 2\log^2(1-z)\right) + \mathcal{O}(\epsilon^3)\Bigg) \end{aligned} \tag{C.20}$$

$$BC_4^{\mathrm{R}} = \Bigg(\frac{1}{\epsilon} + 2 - \log(z) + \epsilon\left(-\frac{7\pi^2}{12} + 4 - 2\log(z) + \frac{\log^2(z)}{2}\right) + \frac{\epsilon^2}{12}\left(-28\zeta_3 - 14\pi^2 + 96 + \left(7\pi^2 - 48\right)\log(z) + 12\log^2(z) - 2\log^3(z)\right) + \mathcal{O}(\epsilon^3)\Bigg) \tag{C.21}$$

$$\begin{aligned} BC_5^{\mathrm{R}} = \Bigg(&- \frac{3H_{0,0,1} - 3H_{0,1,1} + H_0^3 - 2\pi^2 H_0}{3(1-z)} \\ &- \frac{\epsilon}{180(1-z)}\Big(H_0^2\left(180H_{0,1} + 180\pi^2\right) - 120\pi^2 H_{0,1} - 360H_{0,0,1} + 360H_{0,1,1} \\ &\quad + H_1\left(360H_{0,1,1} - 360H_{0,0,1}\right) - 900H_{0,0,0,1} + 1080H_{0,0,1,1} \\ &\quad - 1260H_{0,1,1,1} - 45H_0^4 + \left(-120H_1 - 120\right)H_0^3 \\ &\quad + \left(240\pi^2 H_1 + 240\pi^2\right)H_0 + 41\pi^4\Big) \\ &+ \mathcal{O}(\epsilon)\Bigg) \end{aligned} \tag{C.22}$$

$$\begin{aligned} BC_6^{\mathrm{R}} = \Bigg(&\frac{1}{\epsilon} + 2 + \frac{\epsilon}{12}\left(48 - 7\pi^2\right) + \epsilon^2\left(8 - \frac{7\pi^2}{6} - \frac{7\zeta_3}{3}\right) \\ &\epsilon^3\left(16 - \frac{7\pi^2}{3} + \frac{73\pi^4}{1440} - \frac{14\zeta_3}{3}\right) + \mathcal{O}(\epsilon^4)\Bigg) \end{aligned} \tag{C.23}$$

$$BC_7^{\mathrm{R}} = \left(-\frac{\pi^2}{2} - 3\epsilon\zeta_3 + \frac{5\pi^4\epsilon^2}{72} + \mathcal{O}(\epsilon^3)\right) \tag{C.24}$$

$$BC_8^{\mathrm{R}} = \left(-\frac{\log^2(z)}{2(1-z)} - \frac{\epsilon\left(4\pi^2\log(z) - \log^3(z)\right)}{6(1-z)} + \mathcal{O}(\epsilon^2)\right) \tag{C.25}$$

$$\begin{aligned} BC_9^{\mathrm{R}} = \Bigg(&\frac{1}{3(1-z)}\Big(H_0\left(6H_{0,1} + \pi^2\right) + H_1\left(\pi^2 - 6H_{0,1}\right) - 12H_{0,0,1} \\ &+ 18H_{0,1,1} - H_0^3 - 3H_1H_0^2 + H_1^3 - 18\zeta_3\Big) + \mathcal{O}(\epsilon)\Bigg) \end{aligned} \tag{C.26}$$

$$
\begin{aligned}
BC_{10}^{\mathrm{R}} = \Bigg(& -\frac{1}{6(1-z)}\Big(6H_{0,0,1}-6H_{0,1,1}+3H_0^2(1-z)-3H_1^2(1-z)-6H_1(1-z)+2H_0^3-4\pi^2H_0 \\
& \quad -3\left(2+\pi^2\right)(1-z)\Big) \\
& -\frac{\epsilon}{180(1-z)}\Big(H_1\left(360H_{0,0,1}-360H_{0,1,1}-60\left(\pi^2-9\right)(1-z)\right)+180H_0^2\left(-H_{0,1}-\pi^2\right) \\
& \quad +120\pi^2H_{0,1}+360H_{0,0,1}-360H_{0,1,1}+900H_{0,0,0,1}-1080H_{0,0,1,1}+1260H_{0,1,1,1}+ \\
& \quad H_0^3\left(120H_1-30z+150\right)+H_0\left(120\pi^2(z-3)-240\pi^2H_1\right)+30H_1^3(1-z)+90H_1^2(1-z) \\
& \quad +45H_0^4+180z\zeta_3+60\pi^2z-900z-180\zeta_3-41\pi^4-60\pi^2+900\Big) \\
& +\mathcal{O}(\epsilon^2)\Bigg)
\end{aligned}
\tag{C.27}
$$

$$
\begin{aligned}
BC_{11}^{\mathrm{R}} = \Bigg(& \frac{1}{\epsilon}+\frac{1}{6(1-z)}\Big(-6(z+1)H_{0,1}+6H_{0,0,1}-6H_{0,1,1}-6H_0^2z-4H_0\left(-3z+\pi^2+3\right) \\
& \quad +3H_1^2(z-1)+18H_1(z-1)+2H_0^3+2\left(9(z-1)+\pi^2(2z-1)\right)\Big) \\
& -\frac{\epsilon}{180(1-z)}\Big(H_0^2\left(180H_{0,1}+180H_1(z+1)+180\left(z+\pi^2+1\right)\right) \\
& \quad +H_0\left(-360(z+1)H_{0,1}+240\pi^2H_1-120\pi^2(z-3)\right)-120\left(\pi^2-3(z+1)\right)H_{0,1} \\
& \quad +180(3z+1)H_{0,0,1}-360(2z+1)H_{0,1,1} \\
& \quad +H_1\left(360(z+1)H_{0,1}-360H_{0,0,1}+360H_{0,1,1}-60\left(\left(2\pi^2-9\right)z+9\right)\right) \\
& \quad -900H_{0,0,0,1}+1080H_{0,0,1,1}-1260H_{0,1,1,1}+H_0^3\left(60(z-2)-120H_1\right) \\
& \quad +30H_1^3(z-1)+90H_1^2(z-1)-45H_0^4+180(9z+2\zeta(3)-9) \\
& \quad +\pi^2(45-165z)+41\pi^4\Big) \\
& +\mathcal{O}(\epsilon^2)\Bigg)
\end{aligned}
\tag{C.28}
$$

$$
\begin{aligned}
BC_{12}^{\mathrm{R}} = \Bigg(& -\frac{12\zeta_3+\log^3(1-z)}{6(1-z)}-\frac{\epsilon}{90(1-z)}\Big(2\pi^4+\zeta_3(-180\log(1-z)-360) \\
& \quad +15\pi^2\log^2(1-z)-30\log^3(1-z)\Big)+\mathcal{O}(\epsilon^2)\Bigg)
\end{aligned}
\tag{C.29}
$$

$$
\begin{aligned}
BC^{\mathrm{R}}_{13} = \Bigg(&- \frac{2}{3(1-z)^2}\Big(H_1\left(-24H_{0,1}-8\pi^2\right) - 12H_{0,0,1} + 12H_{0,1,1} \\
&- 2H_0^3 + \left(24H_1^2+2\pi^2\right)H_0 + 16H_1^3 - 48\zeta_3\Big) + \mathcal{O}(\epsilon)\Bigg)
\end{aligned}
\tag{C.30}
$$

$$
\begin{aligned}
BC^{\mathrm{R}}_{14} =\Bigg(&- \frac{1}{\epsilon}\,\frac{1}{(1-z)^3 z}\Big(H_1 z\left(z^2-4z-1\right) + H_0\left(z^3-4z^2+z-2\right) + 2(z-1)\Big) \\
&- \frac{1}{6(1-z)^3 z}\Big(6\left(z^3-4z^2+z+2\right)H_{0,1} + 24zH_{0,0,1} - 24zH_{0,1,1} \\
&\quad + H_1\left(48zH_{0,1} + 4\left(-6z^3+24z^2+\left(15+4\pi^2\right)z+3\right)\right) \\
&\quad + H_0\left(-48H_1^2 z - 12H_1(z-1)^2 z - 24z^3 + 96z^2 - 4\pi^2 z + 48\right) \\
&\quad - 3H_1^2\left(3z^3-4z^2+3z+2\right) + 4H_0^3 z - 12H_0^2(z-1) \\
&\quad - 32H_1^3 z + \pi^2\left(z^3-4z^2+17z-10\right) + 24(z(4\zeta_3-3)+2)\Big) + \mathcal{O}(\epsilon)\Bigg)
\end{aligned}
\tag{C.31}
$$

$$
\begin{aligned}
BC^{\mathrm{R}}_{15} = \Bigg(&- \frac{1}{3(1-z)}\Big(H_1\left(12H_{0,1}+4\pi^2\right) + 6H_{0,0,1} - 6H_{0,1,1} \\
&+ H_0^3 + \left(-12H_1^2-\pi^2\right)H_0 - 8H_1^3 + 24\zeta_3\Big) + \mathcal{O}(\epsilon)\Bigg)
\end{aligned}
\tag{C.32}
$$

$$
\begin{aligned}
BC^{\mathrm{R}}_{16} =\Bigg(&\frac{1}{\epsilon}\,\frac{1}{(1-z)^2}\Big(H_0^2 - H_1^2\Big) \\
&+ \frac{1}{3(1-z)^2}\Big(6H_{0,0,1} - 6H_{0,1,1} - 2H_1^3 + 6H_1^2 + 2\pi^2 H_1 - 6H_0^2 + 3\pi^2 H_0 + 36\zeta_3\Big) + \mathcal{O}(\epsilon)\Bigg)
\end{aligned}
\tag{C.33}
$$

Appendix D

Master Integrals

Following ref. [41], here we present the analytic expressions for masters for completeness.

D.1 Master Integrals in Double-Virtual Contribution

Here we present the master integrals as an expansion in ϵ:

$$
\begin{aligned}
M_1^V(x,\epsilon) = \frac{1}{\epsilon^2}&\left\{x^2+4x^3+10x^4+\mathcal{O}\left(x^5\right)\right\} \\
&+\frac{1}{\epsilon}\left\{x^2(2-2\log x)+x^3(4-8\log x)+x^4(2-20\log x)+\mathcal{O}\left(x^5\right)\right\} \\
&+x^2\left(\zeta_2+2\log^2 x-4\log x+3\right)+x^3\left(4\zeta_2+8\log^2 x-8\log x+4\right) \\
&+x^4\left(10\zeta_2+20\log^2 x-4\log x+2\right)+\mathcal{O}\left(x^5\right) \\
&+\epsilon\left\{x^2\left(2\zeta_2-\frac{2\zeta_3}{3}-2\zeta_2\log x-\frac{4}{3}\log^3 x+4\log^2 x-6\log x+4\right)\right. \\
&\qquad+x^3\left(4\zeta_2-\frac{8\zeta_3}{3}-8\zeta_2\log x-\frac{16}{3}\log^3 x+8\log^2 x-8\log x+4\right) \\
&\left.+x^4\left(2\zeta_2-\frac{20\zeta_3}{3}-20\zeta_2\log x-\frac{40}{3}\log^3 x+4\log^2 x-4\log x+2\right)+\mathcal{O}\left(x^5\right)\right\} \\
&+\mathcal{O}\left(\epsilon^2\right)
\end{aligned}
$$

$$
\begin{aligned}
M_2^V(x,\epsilon) &= \frac{1}{\epsilon^2}\left\{x + 2x^2 + 3x^3 + \mathcal{O}\left(x^4\right)\right\} \\
&+ \frac{1}{\epsilon}\left\{x(3-\log x) + x^2(4-2\log x) + x^3(4-3\log x) + \mathcal{O}\left(x^4\right)\right\} \\
&+ x\left(\frac{\log^2 x}{2} - 3\log x + 7\right) + x^2\left(\log^2 x - 4\log x + 8\right) \\
&+ x^3\left(\frac{3\log^2 x}{2} - 4\log x + 8\right) + \mathcal{O}\left(x^4\right) \\
&+ \epsilon\left\{x\left(-\frac{8\zeta_3}{3} - \frac{1}{6}\log^3 x + \frac{3\log^2 x}{2} - 7\log x + 15\right)\right. \\
&\qquad + x^2\left(-\frac{16\zeta_3}{3} - \frac{1}{3}\log^3 x + 2\log^2 x - 8\log x + 16\right) \\
&\quad \left. + x^3\left(-8\zeta_3 - \frac{1}{2}\log^3 x + 2\log^2 x - 8\log x + 16\right) + \mathcal{O}\left(x^4\right)\right\} + \mathcal{O}\left(\epsilon^2\right)
\end{aligned}
$$

$$
\begin{aligned}
M_3^V(x,\epsilon) &= \frac{1}{\epsilon^2}\left\{x + 2x^2 + 3x^3 + \mathcal{O}\left(x^4\right)\right\} \\
&+ \frac{1}{\epsilon}\left\{x(3-\log x) + 2x^2 + x^3(3\log x - 1) + \mathcal{O}\left(x^4\right)\right\} \\
&+ x\left(\frac{\log^2 x}{2} - 3\log x + 7\right) + x^2\left(-2\zeta_2 - 2\log^2 x + 2\log x + 4\right) \\
&+ x^3\left(-6\zeta_2 - \frac{15}{2}\log^2 x + 4\log x + \frac{13}{2}\right) + \mathcal{O}\left(x^4\right) \\
&+ \epsilon\left\{x\left(-\frac{8\zeta_3}{3} - \frac{1}{6}\log^3 x + \frac{3\log^2 x}{2} - 7\log x + 15\right)\right. \\
&\qquad + x^2\left(-4\zeta_2 - \frac{28\zeta_3}{3} + 4\zeta_2\log x + 2\log^3 x - 5\log^2 x + 2\log x + 10\right) \\
&\qquad + x^3\left(-3\zeta_2 - 20\zeta_3 + 12\zeta_2\log x + \frac{13\log^3 x}{2} - 5\log^2 x - 7\log x + \frac{73}{4}\right) \\
&\qquad \left. + \mathcal{O}\left(x^4\right)\right\} + \mathcal{O}\left(\epsilon^2\right)
\end{aligned}
$$

$$
\begin{aligned}
M_4^V(x,\epsilon) &= \frac{1}{\epsilon^2}\left\{1 + \mathcal{O}\left(x^3\right)\right\} + \frac{1}{\epsilon}\left\{4 + x(2\log x - 2) + x^2(2\log x - 1) + \mathcal{O}\left(x^3\right)\right\} \\
&+ (12-\zeta_2) + x\left(-2\zeta_2 - \log^2 x + 6\log x - 6\right) \\
&+ x^2\left(-2\zeta_2 - \log^2 x + \log x + \frac{3}{2}\right) + \mathcal{O}\left(x^3\right) + \epsilon\left\{\left(-4\zeta_2 - \frac{14\zeta_3}{3} + 32\right)\right. \\
&\qquad + x\left(-4\zeta_2 - 4\zeta_3 + \frac{\log^3 x}{3} - 3\log^2 x + 14\log x - 14\right) \\
&\quad \left. + x^2\left(-4\zeta_3 + \frac{\log^3 x}{3} - \frac{\log^2 x}{2} - \frac{3\log x}{2} + \frac{27}{4}\right) + \mathcal{O}\left(x^3\right)\right\} + \mathcal{O}\left(\epsilon^2\right)
\end{aligned}
$$

$$M_5^V(x,\epsilon) = -\log^2 x + \mathcal{O}\left(x^3\right) + \epsilon\left\{\left(6\zeta_3 + 2\zeta_2\log x + \log^3 x\right) + x\left(6\log^2 x + 4\log x - 12\right)\right.$$
$$\left. + x^2\left(3\log^2 x - 5\log x + \frac{9}{2}\right) + \mathcal{O}\left(x^3\right)\right\} + \mathcal{O}\left(\epsilon^2\right)$$

$$M_6^V(x,\epsilon) = \frac{1}{\epsilon}\left\{-\log x + 2x\log x - 2x^2\log x + \mathcal{O}\left(x^3\right)\right\}$$
$$+ \left(\zeta_2 + \frac{\log^2 x}{2}\right) + x\left(-2\zeta_2 + 8\log x - 4\right)$$
$$+ x^2\left(2\zeta_2 - 18\log x + 10\right) + \mathcal{O}\left(x^3\right) + \epsilon\left\{\left(8\zeta_3 - \frac{\log^3 x}{6}\right)\right.$$
$$+ x\left(-8\zeta_2 - 22\zeta_3 - 2\zeta_2\log x - \frac{2}{3}\log^3 x - 3\log^2 x + 6\log x - 10\right)$$
$$+ x^2\left(18\zeta_2 + 22\zeta_3 + 2\zeta_2\log x + \frac{2\log^3 x}{3} - \frac{3\log^2 x}{2} - \frac{99\log x}{2} + \frac{247}{4}\right)$$
$$\left. + \mathcal{O}\left(x^3\right)\right\} + \mathcal{O}\left(\epsilon^2\right)$$

$$M_7^V(x,\epsilon) = \frac{1}{\epsilon^2}\left\{\frac{1}{2} + \mathcal{O}\left(x^3\right)\right\} + \frac{1}{\epsilon}\left\{\frac{5}{2} + x(2\log x - 2) + x^2(2\log x - 1) + \mathcal{O}\left(x^3\right)\right\}$$
$$+ \left(\frac{\zeta_2}{2} + \frac{19}{2}\right) + x\left(4\zeta_3 + 2\zeta_2\log x + \frac{\log^3 x}{6} - \frac{5\log^2 x}{2} + 9\log x - 9\right)$$
$$+ x^2\left(8\zeta_3 + 4\zeta_2\log x + \frac{\log^3 x}{3} - \frac{5\log^2 x}{2} + \frac{7\log x}{2} - \frac{15}{4}\right) + \mathcal{O}\left(x^3\right)$$
$$+ \epsilon\left\{\left(\frac{5\zeta_2}{2} - \frac{13\zeta_3}{3} + \frac{65}{2}\right) + x\left(-\zeta_2 + \zeta_3 - 11\zeta_4 - \frac{7}{2}\zeta_2\log^2 x + 5\zeta_2\log x\right.\right.$$
$$\left. - 11\zeta_3\log x - \frac{5}{24}\log^4 x + \frac{13\log^3 x}{6} - \frac{21\log^2 x}{2} + 29\log x - 29\right)$$
$$+ x^2\left(\frac{7\zeta_2}{2} - 19\zeta_3 - 22\zeta_4 - 7\zeta_2\log^2 x - 5\zeta_2\log x - 22\zeta_3\log x - \frac{5}{12}\log^4 x\right.$$
$$\left.\left. + \frac{4\log^3 x}{3} - 3\log^2 x + 2\log x + 7\right) + \mathcal{O}\left(x^3\right)\right\} + \mathcal{O}\left(\epsilon^2\right)$$

$$M_8^V(x,\epsilon) = \frac{\log^2 x}{4x} - \frac{\log^2 x}{2} + \frac{1}{4}x\log^2 x + \mathcal{O}\left(x^2\right) + \epsilon\left\{\frac{-\frac{9\zeta_3}{2} - \frac{3}{2}\zeta_2\log x - \frac{5}{12}\log^3 x}{x}\right.$$
$$+ \left(9\zeta_3 + 3\zeta_2\log x + \frac{5\log^3 x}{6} - \frac{\log^2 x}{2} - 3\log x + 5\right)$$
$$\left. + x\left(-\frac{9\zeta_3}{2} - \frac{3}{2}\zeta_2\log x - \frac{5}{12}\log^3 x + \frac{3\log^2 x}{4} + \frac{23\log x}{4} - \frac{79}{8}\right) + \mathcal{O}\left(x^2\right)\right\}$$
$$+ \mathcal{O}\left(\epsilon^2\right)$$

$$M_9^V(x,\epsilon) = -6\zeta_3 + x\left(2\log^2 x - 6\log x + 6\right) + x^2\left(\log^2 x - \frac{3\log x}{2} + \frac{3}{4}\right) + \mathcal{O}\left(x^3\right)$$

$$+\epsilon\left\{(-12\zeta_3 - 9\zeta_4) + x\left(6\zeta_2 - 4\zeta_2\log x - 2\log^3 x + 9\log^2 x - 22\log x + 28\right)\right.$$

$$\left.+x^2\left(\frac{3\zeta_2}{2} - 2\zeta_2\log x - \log^3 x - \frac{23\log^2 x}{4} + \frac{61\log x}{4} - \frac{25}{4}\right) + \mathcal{O}\left(x^3\right)\right\}$$

$$+\mathcal{O}\left(\epsilon^2\right)$$

$$M_{10}^V(x,\epsilon) = \frac{1}{\epsilon}\left\{-\frac{1}{2}x\log^2 x - x^2\log^2 x - \frac{3}{2}x^3\log^2 x + \mathcal{O}\left(x^4\right)\right\}$$

$$+x\left(3\zeta_3 + \zeta_2\log x + \frac{5\log^3 x}{6} - \frac{\log^2 x}{2}\right)$$

$$+x^2\left(6\zeta_3 + 2\zeta_2\log x + \frac{5\log^3 x}{3} + \log^2 x + 2\log x - 4\right)$$

$$+x^3\left(9\zeta_3 + 3\zeta_2\log x + \frac{5\log^3 x}{2} + \frac{7\log^2 x}{2} + \frac{7\log x}{2} - \frac{15}{2}\right) + \mathcal{O}\left(x^4\right)$$

$$+\epsilon\left\{x\left(3\zeta_3 + \frac{5\zeta_4}{4} - 2\zeta_2\log^2 x + \zeta_2\log x - 4\zeta_3\log x - \frac{17}{24}\log^4 x\right.\right.$$

$$\left.+\frac{5\log^3 x}{6} - \frac{\log^2 x}{2}\right) + x^2\left(-2\zeta_2 - 6\zeta_3 + \frac{5\zeta_4}{2} - 4\zeta_2\log^2 x - 2\zeta_2\log x\right.$$

$$\left.-8\zeta_3\log x - \frac{17}{12}\log^4 x - \frac{5\log^3 x}{3} - 2\log^2 x + 8\log x - 4\right)$$

$$+x^3\left(-\frac{7\zeta_2}{2} - 21\zeta_3 + \frac{15\zeta_4}{4} - 6\zeta_2\log^2 x - 7\zeta_2\log x - 12\zeta_3\log x\right.$$

$$\left.\left.-\frac{17}{8}\log^4 x - \frac{35\log^3 x}{6} - \frac{23\log^2 x}{4} + \frac{23\log x}{4} + \frac{21}{2}\right) + \mathcal{O}\left(x^4\right)\right\} + \mathcal{O}\left(\epsilon^2\right)$$

$$
\begin{aligned}
M_{11}^V(x,\epsilon) = &\frac{1}{\epsilon^2}\left\{\frac{1}{2}+\mathcal{O}\left(x^3\right)\right\}+\frac{1}{\epsilon}\left\{\left(\frac{1}{2}-\log x\right)-2x-x^2+\mathcal{O}\left(x^3\right)\right\}\\
&+\left(\frac{\zeta_2}{2}+\frac{\log^2 x}{2}-3\log x-\frac{5}{2}\right)+x\left(-4\zeta_3+\frac{\log^3 x}{3}-2\right)\\
&+x^2\left(-8\zeta_3+\frac{2\log^3 x}{3}+2\log^2 x-6\log x+7\right)+\mathcal{O}\left(x^3\right)\\
&+\epsilon\left\{\left(\frac{5\zeta_2}{2}+\frac{8\zeta_3}{3}-\frac{1}{6}\log^3 x+\frac{3\log^2 x}{2}-7\log x-\frac{35}{2}\right)+x\left(2\zeta_2-4\zeta_3\right.\right.\\
&\left.+4\zeta_4-\zeta_2\log^2 x+2\zeta_3\log x-\frac{5}{12}\log^4 x+\frac{\log^3 x}{3}+\log^2 x-2\log x-4\right)\\
&\quad+x^2\left(7\zeta_2-4\zeta_3+8\zeta_4-2\zeta_2\log^2 x-4\zeta_2\log x+4\zeta_3\log x-\frac{5}{6}\log^4 x\right.\\
&\qquad\left.\left.-\frac{8\log^3 x}{3}+\frac{11\log^2 x}{2}-\frac{9\log x}{2}+\frac{13}{4}\right)+\mathcal{O}\left(x^3\right)\right\}+\mathcal{O}\left(\epsilon^2\right)
\end{aligned}
$$

$$
\begin{aligned}
M_{12}^V(x,\epsilon) = &\frac{1}{\epsilon^2}\left\{\frac{1}{8}+\mathcal{O}\left(x^3\right)\right\}+\frac{1}{\epsilon}\left\{\frac{9}{16}-\frac{x}{2}+x^2\left(-\frac{3\log x}{2}-\frac{1}{4}\right)+\mathcal{O}\left(x^3\right)\right\}\\
&+\left(\frac{\zeta_2}{8}+\frac{63}{32}\right)-\frac{5x}{2}\\
&+x^2\left(-2\zeta_3-\zeta_2\log x-\frac{1}{12}\log^3 x+\frac{15\log^2 x}{8}-\frac{43\log x}{8}-\frac{21}{16}\right)\\
&+\mathcal{O}\left(x^3\right)+\epsilon\left\{\left(\frac{9\zeta_2}{16}-\frac{13\zeta_3}{12}+\frac{405}{64}\right)+\left(-\frac{\zeta_2}{2}-\frac{19}{2}\right)x\right.\\
&+x^2\left(-\frac{9\zeta_2}{8}+\frac{5\zeta_3}{4}+\frac{11\zeta_4}{2}+\frac{7}{4}\zeta_2\log^2 x-\frac{11}{4}\zeta_2\log x+\frac{11}{2}\zeta_3\log x+\frac{5\log^4 x}{48}\right.\\
&\quad\left.\left.-\frac{37\log^3 x}{24}+\frac{101\log^2 x}{16}-\frac{245\log x}{16}-\frac{219}{32}\right)+\mathcal{O}\left(x^3\right)\right\}+\mathcal{O}\left(\epsilon^2\right)
\end{aligned}
$$

$$
\begin{aligned}
M_{13}^V(x,\epsilon) = &\frac{1}{\epsilon^2}\left\{1+\mathcal{O}\left(x^3\right)\right\}+\frac{1}{\epsilon}\left\{4+x(4\log x-4)+x^2(4\log x-2)+\mathcal{O}\left(x^3\right)\right\}\\
&+(12-\zeta_2)+x\left(-4\zeta_2-2\log^2 x+12\log x-12\right)\\
&+x^2\left(-4\zeta_2+2\log^2 x-6\log x+7\right)+\mathcal{O}\left(x^3\right)+\epsilon\left\{\left(-4\zeta_2-\frac{14\zeta_3}{3}+32\right)\right.\\
&\qquad+x\left(-8\zeta_2-8\zeta_3+\frac{2\log^3 x}{3}-6\log^2 x+28\log x-28\right)\\
&\quad+x^2\left(8\zeta_2-8\zeta_3-8\zeta_2\log x-\frac{10}{3}\log^3 x+11\log^2 x-19\log x+\frac{43}{2}\right)\\
&\qquad\left.+\mathcal{O}\left(x^3\right)\right\}+\mathcal{O}\left(\epsilon^2\right)
\end{aligned}
$$

$$
\begin{aligned}
M_{14}^{V}(x,\epsilon) = & \left(9\zeta_4 + 2\zeta_2 \log^2 x + 8\zeta_3 \log x + \frac{\log^4 x}{24}\right) \\
& + x\left(-18\zeta_4 - 4\zeta_2 \log^2 x - 16\zeta_3 \log x - \frac{1}{12}\log^4 x + 3\log^2 x - 18\right) \\
& + x^2\left(18\zeta_4 + 4\zeta_2 \log^2 x + 16\zeta_3 \log x + \frac{\log^4 x}{12} - \frac{13\log^2 x}{4} - 8\log x + \frac{351}{8}\right) \\
& + \mathcal{O}\left(x^3\right) \\
& + \epsilon\left\{\left(-12\zeta_2\zeta_3 + 7\zeta_5 - \frac{5}{2}\zeta_2 \log^3 x - \frac{13}{2}\zeta_3 \log^2 x - 10\zeta_4 \log x - \frac{1}{20}\log^5 x\right)\right. \\
& + x\left(24\zeta_2\zeta_3 - 38\zeta_3 - 54\zeta_4 - 14\zeta_5 + 5\zeta_2 \log^3 x - 12\zeta_2 \log^2 x + 13\zeta_3 \log^2 x\right. \\
& - 2\zeta_2 \log x - 48\zeta_3 \log x + 20\zeta_4 \log x + \frac{\log^5 x}{10} - \frac{\log^4 x}{4} - \frac{10\log^3 x}{3} + 5\log^2 x \\
& \left. - 4\log x + 38\right) + x^2\left(12\zeta_2 - 24\zeta_2\zeta_3 + \frac{117\zeta_3}{2} + 99\zeta_4 + 14\zeta_5 - 5\zeta_2 \log^3 x\right. \\
& + 22\zeta_2 \log^2 x - 13\zeta_3 \log^2 x - \frac{9}{2}\zeta_2 \log x + 88\zeta_3 \log x - 20\zeta_4 \log x - \frac{1}{10}\log^5 x \\
& \left.\left. + \frac{11\log^4 x}{24} + \frac{11\log^3 x}{3} - \frac{161\log^2 x}{8} - \frac{23\log x}{4} + \frac{873}{16}\right) + \mathcal{O}\left(x^3\right)\right\} + \mathcal{O}\left(\epsilon^2\right)
\end{aligned}
$$

$$
\begin{aligned}
M_{15}^{V}(x,\epsilon) = & \left(-\frac{5\zeta_4}{2} - \frac{1}{2}\zeta_2 \log^2 x\right) + x\left(2\zeta_2 - 2\zeta_2 \log x - \frac{1}{2}\log^3 x + \frac{\log^2 x}{2}\right) \\
& + x^2\left(\frac{\zeta_2}{2} - \zeta_2 \log x - \frac{1}{4}\log^3 x - \frac{7\log^2 x}{8} + 2\log x - 1\right) + \mathcal{O}\left(x^3\right) \\
& + \epsilon\left\{\left(-3\zeta_2\zeta_3 + 9\zeta_5 + \frac{1}{2}\zeta_2 \log^3 x - \frac{1}{2}\zeta_3 \log^2 x + 4\zeta_4 \log x\right)\right. \\
& \quad + x\left(-6\zeta_2 + \zeta_3 + 17\zeta_4 + \frac{13}{2}\zeta_2 \log^2 x - \zeta_2 \log x + 9\zeta_3 \log x + \frac{5\log^4 x}{8}\right. \\
& \qquad \left. - \frac{5\log^3 x}{6} + \frac{\log^2 x}{2} - 2\log x + 7\right) \\
& \quad + x^2\left(-\frac{57\zeta_2}{4} + \frac{41\zeta_3}{4} + \frac{17\zeta_4}{2} + \frac{13}{4}\zeta_2 \log^2 x + \frac{47}{4}\zeta_2 \log x + \frac{9}{2}\zeta_3 \log x\right. \\
& \left.\left. + \frac{5\log^4 x}{16} + \frac{77\log^3 x}{24} - \frac{33\log^2 x}{16} - \frac{21\log x}{8} - \frac{245}{32}\right) + \mathcal{O}\left(x^3\right)\right\} + \mathcal{O}\left(\epsilon^2\right)
\end{aligned}
$$

$$
\begin{aligned}
M_{16}^{V}(x,\epsilon) = &\frac{1}{\epsilon}\left\{-\frac{\log^2 x}{2}+\mathcal{O}\left(x^3\right)\right\}+\left(3\zeta_3+\zeta_2\log x+\frac{\log^3 x}{3}-\log^2 x\right)\\
&+x\left(-\log^3 x+2\log^2 x+2\log x-4\right)\\
&+x^2\left(-\log^3 x+\log^2 x-\frac{\log x}{2}+\frac{1}{2}\right)+\mathcal{O}\left(x^3\right)\\
&+\epsilon\left\{\left(6\zeta_3+\frac{5\zeta_4}{4}-\frac{1}{2}\zeta_2\log^2 x+2\zeta_2\log x-\zeta_3\log x-\frac{1}{8}\log^4 x+\frac{2\log^3 x}{3}\right.\right.\\
&\left.-2\log^2 x\right)+x\left(-2\zeta_2-12\zeta_3+3\zeta_2\log^2 x-4\zeta_2\log x+6\zeta_3\log x+\frac{7\log^4 x}{6}\right.\\
&\left.-\frac{7\log^3 x}{3}+2\log^2 x+6\log x-8\right)+x^2\left(\frac{\zeta_2}{2}-6\zeta_3+3\zeta_2\log^2 x-2\zeta_2\log x\right.\\
&\left.\left.+6\zeta_3\log x+\frac{7\log^4 x}{6}+\frac{17\log^3 x}{6}+\frac{\log^2 x}{2}-\frac{73\log x}{4}+19\right)+\mathcal{O}\left(x^3\right)\right\}\\
&+\mathcal{O}\left(\epsilon^2\right)
\end{aligned}
$$

$$
\begin{aligned}
M_{17}^{V}(x,\epsilon) = &\frac{1}{\epsilon}\left\{\left(-12\zeta_3-4\zeta_2\log x+\frac{2\log^3 x}{3}\right)+x(16-8\log x)+x^2(2\log x-2)\right.\\
&\left.+\mathcal{O}\left(x^3\right)\right\}+\left(-32\zeta_4-2\zeta_2\log^2 x-20\zeta_3\log x-\frac{5}{6}\log^4 x\right)\\
&+x\left(24\zeta_2-8\log^2 x+32\right)+x^2\left(-2\zeta_2+26\log x-44\right)+\mathcal{O}\left(x^3\right)\\
&+\epsilon\left\{\left(60\zeta_2\zeta_3-44\zeta_5+\frac{14}{3}\zeta_2\log^3 x+16\zeta_3\log^2 x+9\zeta_4\log x+\frac{17\log^5 x}{30}\right)\right.\\
&+x\left(-16\zeta_2+168\zeta_3+24\zeta_2\log x+18\log^3 x-44\log^2 x+80\log x-144\right)\\
&+x^2\left(-40\zeta_2-22\zeta_3-2\zeta_2\log x-\frac{1}{6}\log^3 x+\frac{33\log^2 x}{2}+23\log x-\frac{233}{2}\right)\\
&\left.+\mathcal{O}\left(x^3\right)\right\}+\mathcal{O}\left(\epsilon^2\right)
\end{aligned}
$$

D.2 Master Integrals in Real-Virtual Contribution

Here we present the master integrals as an expansion in ϵ:

$$
\begin{aligned}
M_{1}^{R}(r,z,\epsilon) = &\frac{1}{\epsilon}\left\{r+\mathcal{O}\left(r^2\right)\right\}+r(1-\log(r))+\mathcal{O}\left(r^2\right)\\
&+\epsilon\left\{\frac{r}{2}\left(\zeta_2+\log^2(r)-2\log(r)+2\right)+\mathcal{O}\left(r^2\right)\right\}\\
&+\epsilon^2\left\{\frac{r}{6}\left(3\zeta_2-2\zeta_3+(-3\zeta_2-6)\log(r)-\log^3(r)+3\log^2(r)+6\right)+\mathcal{O}\left(r^2\right)\right\}\\
&+\mathcal{O}\left(\epsilon^3\right)
\end{aligned}
$$

$$
\begin{aligned}
M_2^R(r,z,\epsilon) = {} & \frac{1}{\epsilon}\{1\} + (3-\log(\bar{z})) \\
& + \frac{r}{\bar{z}}\left(\log(r)(2\log(\bar{z})+2) - \log^2(r) - \log^2(\bar{z}) - 2\log(\bar{z}) - 2\right) \\
& + \mathcal{O}\left(r^2\right) + \epsilon\left\{\frac{1}{2}\left(-3\zeta_2 + \log^2(\bar{z}) - 6\log(\bar{z}) + 18\right)\right. \\
& + \frac{r}{\bar{z}}\left(-4\zeta_2 + 4\zeta_3 + \log(r)\left(4\zeta_2 + \log^2(\bar{z}) + 2\right) + \log^2(r)(-2\log(\bar{z}) - 1)\right. \\
& \left.\left. + \log^3(r) - 4\zeta_2\log(\bar{z}) + \log^2(\bar{z}) - 2\right) + \mathcal{O}\left(r^2\right)\right\} \\
& + \epsilon^2\left\{\frac{1}{6}\left(-27\zeta_2 - 38\zeta_3 + 9\zeta_2\log(\bar{z}) - \log^3(\bar{z}) + 9\log^2(\bar{z}) - 54\log(\bar{z}) + 162\right)\right. \\
& + \frac{r}{12\bar{z}}\left(-12\zeta_2 - 144\zeta_3 + 24\zeta_4\right. \\
& + \log(r)\left(12\zeta_2 + 48\zeta_3 + 60\zeta_2\log(\bar{z}) + 4\log^3(\bar{z}) + 24\right) \\
& + \log^2(r)\left(-54\zeta_2 - 12\log^2(\bar{z}) - 12\right) + \log^3(r)(16\log(\bar{z}) + 4) - 7\log^4(r) \\
& \left. - 6\zeta_2\log^2(\bar{z}) + 36\zeta_2\log(\bar{z}) - 96\zeta_3\log(\bar{z}) - \log^4(\bar{z}) - 4\log^3(\bar{z}) - 24\right) \\
& \left. + \mathcal{O}\left(r^2\right)\right\} + \mathcal{O}\left(\epsilon^3\right)
\end{aligned}
$$

$$
\begin{aligned}
M_3^R(r,z,\epsilon) = {} & \frac{1}{6\bar{z}}\left(-12\zeta_3 - 3\log^2(r)\log(\bar{z}) + 3\log(r)\log^2(\bar{z}) + \log^3(r) - \log^3(\bar{z})\right) \\
& + \frac{r}{\bar{z}^2}(-2\log(r) - 2(-\log(\bar{z}) - 1)) + \mathcal{O}\left(r^2\right) \\
& + \epsilon\left\{\frac{1}{6\bar{z}}\left(24\zeta_3 - 12\zeta_4 + \log^2(r)\left(-6\zeta_2 - 3\log^2(\bar{z}) + 6\log(\bar{z})\right)\right.\right. \\
& + \log(r)\left(12\zeta_2\log(\bar{z}) + \log^3(\bar{z}) - 6\log^2(\bar{z})\right) + \log^3(r)(3\log(\bar{z}) - 2) \\
& \left. - \log^4(r) - 6\zeta_2\log^2(\bar{z}) + 12\zeta_3\log(\bar{z}) + 2\log^3(\bar{z})\right) \\
& \left. + \frac{r}{\bar{z}^2}\left(2\log(r)(\log(\bar{z}) + 3) + 2\left(2\zeta_2 - \log^2(\bar{z}) - 4\log(\bar{z}) - 2\right)\right) + \mathcal{O}\left(r^2\right)\right\} \\
& + \mathcal{O}\left(\epsilon^2\right)
\end{aligned}
$$

$$
\begin{aligned}
M_4^R(r,z,\epsilon) = {} & \frac{1}{\epsilon}\{1\} + (-\log(z) + i\pi + 2) + \mathcal{O}\left(r^1\right) \\
& + \epsilon\left\{\frac{1}{2}\left(-7\zeta_2 + \log^2(z) - 2i\pi\log(z) - 4\log(z) + 4i\pi + 8\right) + \mathcal{O}\left(r^1\right)\right\} \\
& + \epsilon^2\left\{\frac{1}{12}\left(-84\zeta_2 - 28\zeta_3 + 42\zeta_2\log(z) - 2\log^3(z) + 6i\pi\log^2(z)\right.\right. \\
& \left.\left. + 12\log^2(z) - 24i\pi\log(z) - 48\log(z) - 3i\pi^3 + 48i\pi + 96\right) + \mathcal{O}\left(r^1\right)\right\} \\
& + \mathcal{O}\left(\epsilon^3\right)
\end{aligned}
$$

$$
\begin{aligned}
M_6^R(r,z,\epsilon) = {} & \frac{1}{\epsilon}\{1\} + (2 + i\pi) + r(-2\log(r) - 2i\pi + 2) + \mathcal{O}\left(r^2\right) \\
& + \epsilon\left\{\left(-\frac{7\zeta_2}{2} + 2i\pi + 4\right) + r\left(8\zeta_2 + \log^2(r) - 2\log(r) + 2\right) + \mathcal{O}\left(r^2\right)\right\} \\
& + \mathcal{O}\left(\epsilon^2\right)
\end{aligned}
$$

$$
\begin{aligned}
M_7^R(r,z,\epsilon) = &\left(-3\zeta_2 + \frac{\log^2(r)}{2} + i\pi\log(r)\right) + r(2\log(r) + 2i\pi) + \mathcal{O}\left(r^2\right) \\
&+ \epsilon\left\{\left(-3\zeta_3 - \zeta_2\log(r) - \frac{1}{3}\log^3(r) - \frac{1}{2}i\pi\log^2(r) - \frac{i\pi^3}{3}\right)\right. \\
&\left.\quad + r\left(-8\zeta_2 - \log^2(r) + 2\log(r) + 2i\pi - 4\right) + \mathcal{O}\left(r^2\right)\right\} + \mathcal{O}\left(\epsilon^2\right)
\end{aligned}
$$

$$
\begin{aligned}
M_8^R(r,z,\epsilon) = &\frac{1}{2\bar{z}}\left(2\log(r)\log(z) - \log^2(z) + 2i\pi\log(z)\right) \\
&+ \frac{r}{-z\bar{z}}(2\bar{z}\log(r) + 2(-\log(z) + i\pi\bar{z})) + \mathcal{O}\left(r^2\right) \\
&+ \epsilon\left\{\frac{1}{6\bar{z}}\left(-3\log^2(r)\log(z) - 24\zeta_2\log(z) + \log^3(z) - 3i\pi\log^2(z)\right)\right. \\
&+ \frac{r}{-z\bar{z}}\left(-\bar{z}\log^2(r) + 2\bar{z}\log(r) + \log^2(z) - 2i\pi\log(z) - 2\log(z) - 8\zeta_2\bar{z}\right. \\
&\left.\left.+ 2i\pi\bar{z} - 4\bar{z}\right) + \mathcal{O}\left(r^2\right)\right\} + \epsilon^2\left\{\frac{1}{24\bar{z}}\left(12\zeta_2\log(r)\log(z) + 4\log^3(r)\log(z)\right.\right. \\
&\left.+ 42\zeta_2\log^2(z) - 48\zeta_3\log(z) - \log^4(z) + 4i\pi\log^3(z) - 6i\pi^3\log(z)\right) \\
&+ \frac{r}{-6z\bar{z}}\left(\log(r)\left(6\zeta_2\bar{z} + 12\bar{z}\right) + 2\bar{z}\log^3(r) - 6\bar{z}\log^2(r) + 42\zeta_2\log(z)\right. \\
&- 2\log^3(z) + 6i\pi\log^2(z) + 6\log^2(z) - 12i\pi\log(z) + 12\log(z) - 48\zeta_2\bar{z} \\
&\left.\left.- 24\zeta_3\bar{z} - 3i\pi^3\bar{z} - 12i\pi\bar{z}\right) + \mathcal{O}\left(r^2\right)\right\} + \mathcal{O}\left(\epsilon^3\right)
\end{aligned}
$$

$$
\begin{aligned}
M_{13}^R(r,z,\epsilon) = &\frac{1}{\epsilon}\left\{-\frac{\bar{z}}{2}\right\} + \left(\frac{1}{2}\bar{z}\log(\bar{z}) - \frac{7\bar{z}}{4}\right) + r\left(-2\log(r)\log(\bar{z}) + \log^2(r) + \log^2(\bar{z}) + 2\right) \\
&+ \mathcal{O}\left(r^2\right) + \epsilon\left\{\frac{1}{8}\left(6\zeta_2\bar{z} - 45\bar{z} - 2\bar{z}\log^2(\bar{z}) + 14\bar{z}\log(\bar{z})\right)\right. \\
&+ r\left(-4\zeta_3 + \log(r)\left(-4\zeta_2 - \log^2(\bar{z})\right) + 2\log^2(r)\log(\bar{z}) - \log^3(r)\right. \\
&\left.\left.+ 4\zeta_2\log(\bar{z}) - 2\log(\bar{z}) + 6\right) + \mathcal{O}\left(r^2\right)\right\} + \epsilon^2\left\{\frac{1}{48}\left(126\zeta_2\bar{z} + 152\zeta_3\bar{z}\right.\right. \\
&\left.- 36\zeta_2\bar{z}\log(\bar{z}) - 837\bar{z} + 4\bar{z}\log^3(\bar{z}) - 42\bar{z}\log^2(\bar{z}) + 270\bar{z}\log(\bar{z})\right) \\
&+ \frac{r}{12}\left(-36\zeta_2 - 24\zeta_4 + \log(r)\left(-48\zeta_3 - 60\zeta_2\log(\bar{z}) - 4\log^3(\bar{z})\right)\right. \\
&+ \log^2(r)\left(54\zeta_2 + 12\log^2(\bar{z})\right) - 16\log^3(r)\log(\bar{z}) + 7\log^4(r) + 6\zeta_2\log^2(\bar{z}) \\
&\left.\left.+ 96\zeta_3\log(\bar{z}) + \log^4(\bar{z}) + 12\log^2(\bar{z}) - 72\log(\bar{z}) + 216\right) + \mathcal{O}\left(r^2\right)\right\} + \mathcal{O}\left(\epsilon^3\right)
\end{aligned}
$$

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